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Equilibrium Stacks for a Non-Cooperative Game Defined on a Product of Staircase-Function Continuous and Finite Strategy Spaces

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Abstract: A method of solving a non-cooperative game defined on a product of staircase-function strategy spaces is presented. The spaces can be finite and continuous as well. The method is based on stacking equilibria of "short" non-cooperative games, each defined on an interval where the pure strategy value is constant. In the case of finite non-cooperative games, which factually are multidimensional-matrix games, the equilibria are considered in general terms, so they can be in mixed strategies as well. The stack is any combination (succession) of the respective equilibria of the "short" multidimensional-matrix games. Apart from the stack, there are no other equilibria in this "long" (staircase-function) multidimensional-matrix game. An example of staircase-function quadmatrix game is presented to show how the stacking is fulfilled for a case of when every "short" quadmatrix game has a single pure-strategy equilibrium. The presented method, further "breaking" the initial staircase-function game into a succession of "short" games, is far more tractable than a straightforward approach to solving directly the "long" non-cooperative game would be.

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1. Introduction

Non-cooperative games are applied for rationalizing the distribution of limited resources (e. g., see [23, 4, 26, 15]). A simple case of the non-cooperative game is a finite non-cooperative game, which always has an equilibrium, either in pure or mixed strategies [22, 23, 10, 9]. An infinite or continuous non-cooperative game is far more complicated as, opposed to a finite game, an equilibrium is not always determinable. Moreover, a solution of an infinite game, in which a strategy has an infinite support, is not practically realizable [8, 23, 21, 12, 15]. This is due to a finite number of factual actions of a player. Therefore, any game is approximated to a finite one, which always has an equilibrium [22].

A finite non-cooperative game is easily rendered to a multidimensional-matrix game [13, 17], wherein the pure strategy can be a complex action through time rather than an elementary action [4, 26, 3, 1, 17, 18]. Although the game rendering can be fulfilled regardless of the pure strategy complexity [22, 13], such rendering is impossible if the set of the player's strategies is either infinite or continuous. If the player's pure strategy is a function (commonly, it is a function of time), and every player possesses a finite set of such function-strategies, the rendering results in huge multidimensional payoff matrices. This is a far more complicated finite game, in which the player's payoff is a functional [25, 16, 17, 18]. Regardless of the function-strategy set finiteness, each player's functional maps every set of functions (pure strategies of the players defined on a time interval) into a real value. However, a finite game is not obtained by just breaking (sampling) a time interval, on which the pure strategy is defined, into a set of subintervals, on which the strategy could be approximately considered constant. This is so because of the continuity of possible values of the strategy on a subinterval. The continuity is removed by sampling along the strategy value axis [13, 16]. Then the set of function-strategies becomes finite, and that results in a finite non-cooperative game. The size and properties of such a game strongly depend on both samplings [13, 17].

2. Motivation

The number of factual actions of a player in any game has a natural limit, whichever the form of the pure strategy is [23, 10, 9, 12]. Nevertheless, if the rules of a system which is game-modeled are defined and administered beforehand, the administrator is likely to define (or constrain) the form of the strategies players will use [21, 26, 24, 16]. The most trivial case is when the player's pure strategy is an elementary action whose duration is negligibly short and thus is represented as just a time point. This case is exhaustively studied as bimatrix, trimatrix, and dyadic games [6, 22, 23, 15]. In a more complicated case, the player's pure strategy is a function of time [25, 16], so the player's action is a complex process whose duration cannot be reduced to a time point. A way to appropriately administer the players' actions is to constrain them to staircase functions whose points of discontinuities (breakpoints) have to be the same for all the players [20, 24, 16]. Along with the discrete time, possible values of the player's pure strategy should be discrete as well. Then the game can be represented as a multidimensional-matrix game, in which the player's selection of a pure strategy means using a staircase function on a time interval whereon every pure strategy is defined.

It is easy to get convinced of that the number of the player's pure strategies in the multidimensional-matrix staircase-function game grows immensely as the number of breakpoints ("stair" intervals) or/and the number of possible values of the player's pure strategy increases. For instance, if the number of intervals is 4, and the number of possible values of the player's pure strategy is 5, then there are $5^4 = 625$ possible pure strategies at this player, where every strategy is a 4-interval 5-staircased function of time. Whereas the respective bimatrix 625×625 game still may be solved in a reasonable time, the respective trimatrix $625 \times 625 \times 625$ game appears to be big enough (having 244140625 situations), let alone $625 \times 625 \times 625 \times 625$ quadmatrix game whose number of situations is 152587890625 (more than 152 billion). This trivialized example shows that a finite staircase-function game becomes practically intractable to solve it when there are more than two players. An exclusion is the ultimately trivialized instance, when every player has 2-interval 2-staircased function-strategies. Then the respective 4×4, $4 \times 4 \times 4$, $4 \times 4 \times 4 \times 4$, ..., games can be solved fast enough even for 10 players, although the $\mathop{\times}\limits_{n=1}^{10} 4$ game has 1048576 situations. It is worth noting that it may take no less than 0.4 seconds to solve a $\mathop{\times}\limits_{n=1}^{6} 4$ game on a laptop with an Intel Core i7 processor, whereas a $10 \times 10 \times 10 \times 10$ game is solved at least in 1.1 seconds. When every strategy, say, is a 6-interval 10-staircased function of time, even the respective bimatrix $10^6 \times 10^6$ staircase-function game appears to be intractably gigantic (there is a trillion situations in this game!). This is a simple example of the intractability even for a bimatrix game, let alone finite staircase-function games with three or more players. This means that, instead of rendering a non-cooperative game defined on a product of staircase-function finite spaces to a multidimensional-matrix game, a tractable method of solving it should be suggested.

3. Objective and tasks to be fulfilled

Issuing from the impracticability of rendering a finite non-cooperative game with staircase-function strategies to a multidimensional-matrix game, the objective is to develop a tractable method of solving non-cooperative games defined on a product of staircase-function finite spaces. For achieving the objective, the following five tasks are to be fulfilled:

1. To formalize a non-cooperative game (of any number of players), in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time. Herein, the time can be thought of as it is discrete.

2. To discretize the set of possible values of the player's pure strategy so that the game be defined on a product of staircase-function finite spaces.

3. To formalize a method of solving non-cooperative games defined on a product of staircase-function finite spaces.

4. To consider an example of solving a finite game defined on a product of staircase-function spaces. A special attention should be paid to the computation time.

5. To discuss and conclude on applicability and significance of the method, as well as its possible drawbacks and limitations.

4. A non-cooperative game with staircase-function strategies

Consider a non-cooperative game of N players, $N \in \mathbb{N} \backslash \{1\}$. In this game the player's pure strategy is a function of time. Let each of the players use time-varying strategies defined almost everywhere on interval $[t_1;\ t_2]$ by $t_2 > t_1$. Denote a strategy of the n-th player by $x_n(t)$, $n = \overline{1,\ N}$. These functions are presumed to be bounded, i. e.

$$x_n^{(\min)} \leqslant x_n(t) \leqslant x_n^{(\max)} \ \text{by} \ x_n^{(\min)} < x_n^{(\max)} \tag{1}$$

defined almost everywhere on $[t_1;\ t_2]$. Besides, the square of the function-strategy is presumed to be Lebesgue-integrable. Thus, pure strategies of the player belong to a rectangular functional space of functions of time:

$$X_n = \\ = \left\{ x_n(t)\,,\ t \in [t_1;\ t_2]\,,\ t_1 < t_2 :\ x_n^{(\min)} \leqslant x(t) \leqslant x_n^{(\max)} \text{ by } x_n^{(\min)} < x_n^{(\max)} \right\} \subset \\ \subset \mathbb{L}_2[t_1;\ t_2] \tag{2}$$

is the set of the n-th player's pure strategies, $n = \overline{1,\ N}$.

The player's payoff in situation

$$\{x_n(t)\}_{n=1}^{N} \tag{3}$$

is presumed to be an integral functional [2, 11, 18, 19]. Thus, the n-th player's payoff in situation (3) is

$$K_n\left(\{x_i(t)\}_{i=1}^{N}\right) = \int\limits_{[t_1;\ t_2]} f_n\left(\{x_i(t)\}_{i=1}^{N},\ t\right) d\mu(t) \tag{4}$$

by a function

$$f_n\left(\{x_i(t)\}_{i=1}^{N},\ t\right) \tag{5}$$

of time functions (3) explicitly including time t. Therefore, the continuous N-person game

$$\left\langle \{X_n\}_{n=1}^{N},\ \left\{K_n\left(\{x_i(t)\}_{i=1}^{N}\right)\right\}_{n=1}^{N} \right\rangle \tag{6}$$

is defined on product

$$\mathop{\times}_{n=1}^{N} X_n \subset \mathop{\times}_{n=1}^{N} \mathbb{L}_2\left[t_1;\ t_2\right] \tag{7}$$

of rectangular functional spaces (2) of players' pure strategies.

First, it is presumed that game (6) is administered so that the players are forced to use pure strategies $\{x_i(t)\}_{i=1}^{N}$ such that they change their values for a finite number of times. Denote by M the number of intervals at which the player's pure strategy is constant, where $M \in \mathbb{N} \backslash \{1\}$. Then the player's pure strategy is a staircase function having only M different values. If $\left\{\tau^{(l)}\right\}_{l=1}^{M-1}$ are time points at which the staircase-function strategy changes its value, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(M-1)} < \tau^{(M)} = t_2, \tag{8}$$

then

$$\left\{x_n\left(\tau^{(l)}\right)\right\}_{l=0}^{M} \tag{9}$$

are the values of the n-th player's strategy in a play-off of game (6), $n = \overline{1,\ N}$. The staircase-function strategies are right-continuous [2]:

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x_n\left(\tau^{(l)} + \varepsilon\right) = x_n\left(\tau^{(l)}\right) \text{ for } l = \overline{1,\ M-1} \text{ by } n = \overline{1,\ N}, \tag{10}$$

whereas

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x_n\left(\tau^{(l)} - \varepsilon\right) \neq x_n\left(\tau^{(l)}\right) \text{ for } l = \overline{1,\ M-1} \text{ by } n = \overline{1,\ N}. \tag{11}$$

As an exception,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x_n\left(\tau^{(M)} - \varepsilon\right) = x_n\left(\tau^{(M)}\right), \tag{12}$$

so

$$x_n\left(\tau^{(M-1)}\right) = x_n\left(\tau^{(M)}\right) \ \forall\, n = \overline{1,\ N}.$$

Then constant values (9) by (8) mean that game (6) can be thought of as it is a succession of M continuous games

$$\left\langle \left\{\left[x_n^{(\min)};\ x_n^{(\max)}\right]\right\}_{n=1}^{N},\ \left\{K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right)\right\}_{n=1}^{N} \right\rangle \tag{13}$$

defined on hyperparallelepiped

$$\mathop{\times}_{n=1}^{N} \left[x_n^{(\min)};\ x_n^{(\max)}\right] \tag{14}$$

by

$$\alpha_{nl} = x_n(t) \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \text{ by } n = \overline{1,\ N}$$

$$\forall\, t \in \left[\tau^{(l-1)};\ \tau^{(l)}\right) \ \text{ for } \ l = \overline{1,\ M-1} \ \text{ and } \ \forall\, t \in \left[\tau^{(M-1)};\ \tau^{(M)}\right], \tag{15}$$

where the factual players' payoffs in situation $\{\alpha_{il}\}_{i=1}^{N}$ are

$$K_n\left(\{\alpha_{il}\}_{i=1}^{N}\right) = \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n\left(\{\alpha_{il}\}_{i=1}^{N},\ t\right) d\mu\,(t) \ \ \forall\, l = \overline{1,\ M-1} \tag{16}$$

by

$$K_n\left(\{\alpha_{iM}\}_{i=1}^{N}\right) = \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\{\alpha_{iM}\}_{i=1}^{N},\ t\right) d\mu\,(t) \tag{17}$$

for $n = \overline{1,\ N}$. So, let such game (6) be called staircase [18, 19]. A pure-strategy situation in staircase game (6) is a succession of M situations $\{\alpha_{il}\}_{i=1}^{N}$ in games (13).

Theorem 1. *In a pure-strategy situation of staircase game (6), represented as a succession of M games (13), functional (4) is re-written as an interval-wise sum*

$$K_n\left(\{x_i\,(t)\}_{i=1}^{N}\right) = \\ = \sum_{l=1}^{M-1} \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n\left(\{\alpha_{il}\}_{i=1}^{N},\ t\right) d\mu\,(t) + \\ + \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\{\alpha_{iM}\}_{i=1}^{N},\ t\right) d\mu\,(t). \tag{18}$$

Proof. Situation $\{\alpha_{il}\}_{i=1}^{N}$ is tied to half-interval $\left[\tau^{(l-1)};\ \tau^{(l)}\right)$ by $l = \overline{1,\ M-1}$ and to interval $\left[\tau^{(M-1)};\ \tau^{(M)}\right]$ by $l = M$. Function (5) in this situation is some function of time t. Denote this function by $\psi_{nl}\,(t)$. For situation $\{\alpha_{il}\}_{i=1}^{N}$ function

$$\psi_{nl}\,(t) = 0 \ \ \forall\, t \notin \left[\tau^{(l-1)};\ \tau^{(l)}\right), \tag{19}$$

and for situation $\{\alpha_{iM}\}_{i=1}^{N}$ function

$$\psi_{nM}\,(t) = 0 \ \ \forall\, t \notin \left[\tau^{(M-1)};\ \tau^{(M)}\right]. \tag{20}$$

Therefore,

$$f_n\left(\{x_i\,(t)\}_{i=1}^{N},\ t\right) = \sum_{l=1}^{M} \psi_{nl}\,(t) \tag{21}$$

in a pure-strategy situation $\{x_i\,(t)\}_{i=1}^{N}$ of staircase game (6), by using (19) and (20). Consequently,

$$K_n\left(\{x_i\,(t)\}_{i=1}^{N}\right) = \int\limits_{[t_1;\ t_2]} f_n\left(\{x_i\,(t)\}_{i=1}^{N},\ t\right) d\mu\,(t) =$$

$$= \sum_{l=1}^{M-1} \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} \psi_{nl}(t)\, d\mu(t) + \int\limits_{\left[\tau^{(M-1)};\, \tau^{(M)}\right]} \psi_{nM}(t)\, d\mu(t) =$$

$$= \sum_{l=1}^{M-1} \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} f_n\left(\{\alpha_{il}\}_{i=1}^{N},\, t\right) d\mu(t) +$$

$$+ \int\limits_{\left[\tau^{(M-1)};\, \tau^{(M)}\right]} f_n\left(\{\alpha_{iM}\}_{i=1}^{N},\, t\right) d\mu(t) \tag{22}$$

in a pure-strategy situation $\{x_i(t)\}_{i=1}^{N}$ of staircase game (6). □

In other words, if every equilibrium situation in pure strategies in game (6) on product (7) by conditions (1) — (5) is (or forced to be) of staircase functions satisfying conditions (8) — (12), then this game is equivalent to the succession of M games (13) defined on parallelepiped (14) by (8) — (12) and (15) — (18). In this case game (6) can be represented by the succession of games (13).

Theorem 2. *If each of M games (13) by (8) — (12) and (15) — (18) has a single equilibrium situation in pure strategies, and game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of these games, then the equilibrium situation in pure strategies in game (6) is determined by independently finding pure-strategy equilibria in M games (13), whereupon these equilibria are successively stacked.*

Proof. First, the equivalency means that game (6) has only staircase pure-strategy equilibria. Next, it should be proved that game (6) has a pure-strategy equilibrium situation, which is a successive stack of the M "short" games (13). Let $\left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\right\}_{l=1}^{M}$ be pure-strategy equilibria in games (13) by (8) — (12) and (15) — (18). Then

$$K_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N} \setminus \{\alpha_{nl}^{*}\}\right\} \cup \{\alpha_{nl}\}\right) \leqslant K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)$$
$$\forall\, \alpha_{nl} \in \left[x_n^{(\min)};\, x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{1,\, N} \text{ and } \forall\, l = \overline{1,\, M}. \tag{23}$$

Inequalities (23) are re-written using statements (15) — (18):

$$K_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N} \setminus \{\alpha_{nl}^{*}\}\right\} \cup \{\alpha_{nl}\}\right) =$$

$$= \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} f_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N} \setminus \{\alpha_{nl}^{*}\}\right\} \cup \{\alpha_{nl}\},\, t\right) d\mu(t) \leqslant$$

$$\leqslant \int\limits_{\left[\tau^{(l-1)};\, \tau^{(l)}\right)} f_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N},\, t\right) d\mu(t) = K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)$$

$$\forall\, \alpha_{nl} \in \left[x_n^{(\min)};\, x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{1,\, N} \text{ and } \forall\, l = \overline{1,\, M-1}, \tag{24}$$

$$K_n\left(\left\{\{\alpha^*_{iM}\}_{i=1}^{N}\setminus\{\alpha^*_{nM}\}\right\}\cup\{\alpha_{nM}\}\right)=$$
$$=\int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\left\{\{\alpha^*_{iM}\}_{i=1}^{N}\setminus\{\alpha^*_{nM}\}\right\}\cup\{\alpha_{nM}\},\ t\right)d\mu(t)\leqslant$$
$$\leqslant\int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\{\alpha^*_{iM}\}_{i=1}^{N},\ t\right)d\mu(t)=$$
$$=K_n\left(\{\alpha^*_{iM}\}_{i=1}^{N}\right)\ \ \forall\,\alpha_{nM}\in\left[x_n^{(\min)};\ x_n^{(\max)}\right]\ \text{ and }\ \forall\,n=\overline{1,\ N}. \tag{25}$$

Owing to Theorem 1,

$$\sum_{l=1}^{M}K_n\left(\left\{\{\alpha^*_{il}\}_{i=1}^{N}\setminus\{\alpha^*_{nl}\}\right\}\cup\{\alpha_{nl}\}\right)\leqslant\sum_{l=1}^{M}K_n\left(\{\alpha^*_{il}\}_{i=1}^{N}\right)\ \ \forall\,n=\overline{1,\ N}. \tag{26}$$

Therefore, the successive stack of pure-strategy equilibria $\left\{\{\alpha^*_{il}\}_{i=1}^{N}\right\}_{l=1}^{M}$ is a pure-strategy equilibrium in game (6). Obviously, games (13) can be solved independently, whose equilibria are stacked afterwards to form the pure-strategy equilibrium in game (6). □

In fact, Theorem 2 claims that if each of N "short" games (13) has a single pure-strategy equilibrium, then the solution of N-person game (6) can be determined in a simpler way, by solving games (13) and successively stacking their equilibria. They are solved in parallel (independently), without caring of the succession. However, Theorem 2 does not determine a probability (likelihood) of the case when every "short" game has a single pure-strategy equilibrium. Obviously, the likelihood decays as the number of intervals increases.

Besides, Theorem 2 does not directly imply that the stacked equilibrium in game (6) is single. The question of whether the stacked equilibrium in game (6) is single or not is answered by the following assertion.

Theorem 3. *If each of M games (13) by (8) — (12) and (15) — (18) has a single equilibrium situation in pure strategies, and game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of these games, then the equilibrium situation in pure strategies in game (6) is single being the successive stack of the "short" games equilibria.*

Proof. The pure-strategy equilibrium in game (6) is constructed according to Theorem 2, i. e., it is the successive stack of pure-strategy equilibria $\left\{\{\alpha^*_{il}\}_{i=1}^{N}\right\}_{l=1}^{M}$. Let this equilibrium be referred to as the

$$\left\{\{\alpha^*_{il}\}_{i=1}^{N}\right\}_{l=1}^{M}\text{-stack equilibrium.} \tag{27}$$

Suppose that there is another pure-strategy equilibrium in game (6). Without losing generality, let this equilibrium differ from (27) in just that the first player uses some $\alpha_{1k}^{(0)} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]$ instead of α_{1k}^{*} by some $k \in \left\{\overline{1,\ M}\right\}$. So, this is the

$$\left\{\left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\right\}_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k\}} \cup \left\{\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right\}\right\}\text{-stack equilibrium,}$$

which means that

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k\}} K_1\left(\alpha_{1l},\ \{\alpha_{il}^{*}\}_{i=2}^{N}\right) + K_1\left(\alpha_{1k},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right) \leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k\}} K_1\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) + K_1\left(\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right), \tag{28}$$

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k\}} K_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\setminus\{\alpha_{nl}^{*}\}\right\}\cup\{\alpha_{nl}\}\right) +$$
$$+K_n\left(\left\{\left\{\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right\}\setminus\{\alpha_{nk}^{*}\}\right\}\cup\{\alpha_{nk}\}\right) \leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k\}} K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) + K_n\left(\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right)\ \ \forall\, n=\overline{2,\ N}, \tag{29}$$

i. e.,

$$K_1\left(\alpha_{1l},\ \{\alpha_{il}^{*}\}_{i=2}^{N}\right) \leqslant K_1\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)$$
$$\forall\, \alpha_{1l} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]\ \text{ and }\ \forall\, l \in \left\{\overline{1,\ M}\right\}\setminus\{k\} \tag{30}$$

by

$$K_1\left(\alpha_{1k},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right) \leqslant K_1\left(\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right)\ \ \forall\, \alpha_{1k} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right], \tag{31}$$

and

$$K_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\setminus\{\alpha_{nl}^{*}\}\right\}\cup\{\alpha_{nl}\}\right) \leqslant K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right)$$
$$\forall\, \alpha_{nl} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right]\ \text{ and }\ \forall\, l \in \left\{\overline{1,\ M}\right\}\setminus\{k\}\ \text{ and }\ \forall\, n=\overline{2,\ N} \tag{32}$$

by

$$K_n\left(\left\{\left\{\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right\}\setminus\{\alpha_{nk}^{*}\}\right\}\cup\{\alpha_{nk}\}\right) \leqslant K_n\left(\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right)$$
$$\forall\, \alpha_{nk} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right]\ \text{ and }\ \forall\, n=\overline{2,\ N}. \tag{33}$$

Inequalities (31) and (33) imply that $\left\{\alpha_{1k}^{(0)},\ \{\alpha_{ik}^{*}\}_{i=2}^{N}\right\}$ is a pure-strategy equilibrium at the k-th interval (in the k-th game), which is impossible due to every interval has a single pure-strategy equilibrium. The impossibility of the other pure-strategy equilibrium for the remaining players in such a case is proved symmetrically.

Suppose that the other pure-strategy equilibrium differs from (27) in that the first player uses some $\alpha_{1k_1}^{(0)} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]$ instead of $\alpha_{1k_1}^{*}$ by some $k_1 \in \left\{\overline{1,\ M}\right\}$ and the second player uses some $\alpha_{2k_2}^{(0)} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right]$ instead of $\alpha_{2k_2}^{*}$ by some $k_2 \in \left\{\overline{1,\ M}\right\}$. So, this is the

$$\left\{\left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\right\}_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} \cup \left\{\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \{\alpha_{ik_1}^{*}\}_{i=3}^{N}\right\}\right\}\text{-stack equilibrium} \tag{34}$$

if $k_1 = k_2$, and is the

$$\left\{\left\{\{\alpha_{il}^{*}\}_{i=1}^{N}\right\}_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1,\ k_2\}} \cup\right.$$
$$\left.\cup\left\{\alpha_{1k_1}^{(0)},\ \{\alpha_{ik_1}^{*}\}_{i=2}^{N}\right\} \cup \left\{\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \{\alpha_{ik_2}^{*}\}_{i=3}^{N}\right\}\right\}\text{-stack equilibrium} \tag{35}$$

if $k_1 \neq k_2$. Thus, (34) means that

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_1\left(\alpha_{1l},\ \{\alpha_{il}^{*}\}_{i=2}^{N}\right) + K_1\left(\alpha_{1k_1},\ \alpha_{2k_1}^{(0)},\ \{\alpha_{ik_1}^{*}\}_{i=3}^{N}\right) \leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_1\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) + K_1\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \{\alpha_{ik_1}^{*}\}_{i=3}^{N}\right) \tag{36}$$

and

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_2\left(\alpha_{1l}^{*},\ \alpha_{2l},\ \{\alpha_{il}^{*}\}_{i=3}^{N}\right) + K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1},\ \{\alpha_{ik_1}^{*}\}_{i=3}^{N}\right) \leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_2\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) + K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \{\alpha_{ik_1}^{*}\}_{i=3}^{N}\right) \tag{37}$$

and

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_n\left(\left\{\{\alpha_{il}^{*}\}_{i=1}^{N} \setminus \{\alpha_{nl}^{*}\}\right\} \cup \{\alpha_{nl}\}\right) +$$
$$+K_n\left(\left\{\left\{\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \{\alpha_{ik_1}^{*}\}_{i=3}^{N}\right\} \setminus \{\alpha_{nk_1}^{*}\}\right\} \cup \{\alpha_{nk_1}\}\right) \leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1\}} K_n\left(\{\alpha_{il}^{*}\}_{i=1}^{N}\right) + K_n\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \{\alpha_{ik_1}^{*}\}_{i=3}^{N}\right) \quad \forall\, n = \overline{3,\ N}, \tag{38}$$

i. e., inequalities (30) by $k = k_1$ and inequality

$$K_1\left(\alpha_{1k_1},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^N\right) \leqslant K_1\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^N\right)$$
$$\forall\,\alpha_{1k_1} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right] \tag{39}$$

hold along with (23) for $n = 1$, inequalities

$$K_2\left(\alpha_{1l}^*,\ \alpha_{2l},\ \{\alpha_{il}^*\}_{i=3}^N\right) \leqslant K_2\left(\{\alpha_{il}^*\}_{i=1}^N\right)$$
$$\forall\,\alpha_{2l} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right] \text{ and } \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1\} \tag{40}$$

and inequality

$$K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^N\right) \leqslant K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^N\right)$$
$$\forall\,\alpha_{2k_1} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right] \tag{41}$$

hold along with (23) for $n = 2$, inequalities (32) by $k = k_1$ and inequality

$$K_n\left(\left\{\left\{\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^N\right\} \setminus \left\{\alpha_{nk_1}^*\right\}\right\} \cup \{\alpha_{nk_1}\}\right) \leqslant$$
$$\leqslant K_n\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^N\right)$$
$$\forall\,\alpha_{nk_1} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{3,\ N} \tag{42}$$

hold along with (23). Inequalities (39) — (42) imply that $\left\{\alpha_{1k_1}^{(0)},\ \alpha_{2k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=3}^N\right\}$ is a pure-strategy equilibrium at the k_1-th interval (in the k_1-th game), which is impossible. The same conclusion is valid for a two-person non-cooperative game, where (34), (36), (37), (39), (41) are written by retaining $\left\{\alpha_{ik_1}^*\right\}_{i=3}^N = \emptyset$, $\{\alpha_{il}^*\}_{i=3}^N = \emptyset$, and (38), (42) are omitted. If (35) is true, then

$$\sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2\}} K_1\left(\alpha_{1l},\ \{\alpha_{il}^*\}_{i=2}^N\right) +$$
$$+K_1\left(\alpha_{1k_1},\ \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right) + K_1\left(\alpha_{1k_2},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^*\right\}_{i=3}^N\right) \leqslant$$
$$\leqslant \sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2\}} K_1\left(\{\alpha_{il}^*\}_{i=1}^N\right) +$$
$$+K_1\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^*\right\}_{i=2}^N\right) + K_1\left(\alpha_{1k_2}^*,\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^*\right\}_{i=3}^N\right) \tag{43}$$

and

$$\sum_{l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2\}} K_2\left(\alpha_{1l}^*,\ \alpha_{2l},\ \{\alpha_{il}^*\}_{i=3}^N\right) +$$

$$+K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right)+K_2\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)\leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}} K_2\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right)+$$
$$+K_2\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_2\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right) \tag{44}$$

and

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}} K_n\left(\left\{\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\setminus\left\{\alpha_{nl}^{*}\right\}\right\}\cup\left\{\alpha_{nl}\right\}\right)+$$
$$+K_n\left(\left\{\left\{\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right\}\setminus\left\{\alpha_{nk_1}^{*}\right\}\right\}\cup\left\{\alpha_{nk_1}\right\}\right)+$$
$$+K_n\left(\left\{\left\{\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right\}\setminus\left\{\alpha_{nk_2}^{*}\right\}\right\}\cup\left\{\alpha_{nk_2}\right\}\right)\leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}} K_n\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right)+K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+$$
$$+K_n\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)\quad \forall\, n=\overline{3,\ N}, \tag{45}$$

i. e., inequalities

$$K_1\left(\alpha_{1l},\ \left\{\alpha_{il}^{*}\right\}_{i=2}^{N}\right)\leqslant K_1\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right)$$
$$\forall\,\alpha_{1l}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right]\ \text{ and }\ \forall\, l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\} \tag{46}$$

and inequality

$$K_1\left(\alpha_{1k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_1\left(\alpha_{1k_2},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)\leqslant$$
$$\leqslant K_1\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_1\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)$$
$$\forall\,\alpha_{1k_1}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right]\ \text{ and }\ \forall\,\alpha_{1k_2}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right] \tag{47}$$

hold along with (23) for $n=1$, inequalities

$$K_2\left(\alpha_{1l}^{*},\ \alpha_{2l},\ \left\{\alpha_{il}^{*}\right\}_{i=3}^{N}\right)\leqslant K_2\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right)$$
$$\forall\,\alpha_{2l}\in\left[x_2^{(\min)};\ x_2^{(\max)}\right]\ \text{ and }\ \forall\, l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\} \tag{48}$$

and inequality

$$K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right)+K_2\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)\leqslant$$
$$\leqslant K_2\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_2\left(\alpha_{1k_2}^{*},\ \alpha_{2k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=3}^{N}\right)$$
$$\forall\,\alpha_{2k_1}\in\left[x_2^{(\min)};\ x_2^{(\max)}\right]\ \text{ and }\ \forall\,\alpha_{2k_2}\in\left[x_2^{(\min)};\ x_2^{(\max)}\right] \tag{49}$$

hold along with (23) for $n=2$, inequalities

$$K_n\left(\left\{\{\alpha_{il}^*\}_{i=1}^N \setminus \{\alpha_{nl}^*\}\right\} \cup \{\alpha_{nl}\}\right) \leqslant K_n\left(\{\alpha_{il}^*\}_{i=1}^N\right)$$
$$\forall\, \alpha_{nl} \in \left[x_n^{(\min)};\; x_n^{(\max)}\right] \text{ and } \forall\, l \in \left\{\overline{1,\; M}\right\} \setminus \{k_1,\; k_2\} \text{ and } \forall\, n=\overline{3,\; N} \tag{50}$$

and inequality

$$K_n\left(\left\{\left\{\alpha_{1k_1}^{(0)},\; \{\alpha_{ik_1}^*\}_{i=2}^N\right\} \setminus \{\alpha_{nk_1}^*\}\right\} \cup \{\alpha_{nk_1}\}\right) +$$
$$+K_n\left(\left\{\left\{\alpha_{1k_2}^*,\; \alpha_{2k_2}^{(0)},\; \{\alpha_{ik_2}^*\}_{i=3}^N\right\} \setminus \{\alpha_{nk_2}^*\}\right\} \cup \{\alpha_{nk_2}\}\right) \leqslant$$
$$\leqslant K_n\left(\alpha_{1k_1}^{(0)},\; \{\alpha_{ik_1}^*\}_{i=2}^N\right) + K_n\left(\alpha_{1k_2}^*,\; \alpha_{2k_2}^{(0)},\; \{\alpha_{ik_2}^*\}_{i=3}^N\right)$$
$$\forall\, \alpha_{nk_1} \in \left[x_n^{(\min)};\; x_n^{(\max)}\right] \text{ and } \forall\, \alpha_{nk_2} \in \left[x_n^{(\min)};\; x_n^{(\max)}\right] \text{ and } \forall\, n=\overline{3,\; N} \tag{51}$$

hold along with (23). Plugging $\alpha_{1k_2}=\alpha_{1k_2}^*$ in the left side of inequality (47) and plugging $\alpha_{2k_2}=\alpha_{2k_2}^{(0)}$ in the left side of inequality (49) and plugging $\alpha_{nk_2}=\alpha_{nk_2}^*$ in the left side of inequality (51) for $n=\overline{3,\; N}$ gives inequalities

$$K_1\left(\alpha_{1k_1},\; \{\alpha_{ik_1}^*\}_{i=2}^N\right) \leqslant K_1\left(\alpha_{1k_1}^{(0)},\; \{\alpha_{ik_1}^*\}_{i=2}^N\right) \;\; \forall\, \alpha_{1k_1} \in \left[x_1^{(\min)};\; x_1^{(\max)}\right], \tag{52}$$

$$K_2\left(\alpha_{1k_1}^{(0)},\; \alpha_{2k_1},\; \{\alpha_{ik_1}^*\}_{i=3}^N\right) \leqslant K_2\left(\alpha_{1k_1}^{(0)},\; \{\alpha_{ik_1}^*\}_{i=2}^N\right)$$
$$\forall\, \alpha_{2k_1} \in \left[x_2^{(\min)};\; x_2^{(\max)}\right], \tag{53}$$

$$K_n\left(\left\{\left\{\alpha_{1k_1}^{(0)},\; \{\alpha_{ik_1}^*\}_{i=2}^N\right\} \setminus \{\alpha_{nk_1}^*\}\right\} \cup \{\alpha_{nk_1}\}\right) \leqslant K_n\left(\alpha_{1k_1}^{(0)},\; \{\alpha_{ik_1}^*\}_{i=2}^N\right)$$
$$\forall\, \alpha_{nk_1} \in \left[x_n^{(\min)};\; x_n^{(\max)}\right] \text{ and } \forall\, n=\overline{3,\; N}, \tag{54}$$

which are impossible due to $\left\{\alpha_{1k_1}^{(0)},\; \{\alpha_{ik_1}^*\}_{i=2}^N\right\}$ is not a pure-strategy equilibrium. Therefore, the supposition about (34) and (35) are true is contradictory. The same conclusion is valid for a two-person non-cooperative game, where (35), (43), (44), (47) — (49), (53) are written by retaining $\{\alpha_{ik_2}^*\}_{i=3}^N=\emptyset$, $\{\alpha_{il}^*\}_{i=3}^N=\emptyset$, $\{\alpha_{ik_1}^*\}_{i=3}^N=\emptyset$, and (45), (50), (51), (54) are omitted.

Now, for the case of $N \geqslant 3$, suppose that the other pure-strategy equilibrium differs from (27) in that the first player uses some $\alpha_{1k_1}^{(0)} \in \left[x_1^{(\min)};\; x_1^{(\max)}\right]$ instead of $\alpha_{1k_1}^*$ by some $k_1 \in \left\{\overline{1,\; M}\right\}$, the second player uses some $\alpha_{2k_2}^{(0)} \in \left[x_2^{(\min)};\; x_2^{(\max)}\right]$ instead of $\alpha_{2k_2}^*$ by some $k_2 \in \left\{\overline{1,\; M}\right\}$, and the third player uses some $\alpha_{3k_3}^{(0)} \in \left[x_3^{(\min)};\; x_3^{(\max)}\right]$ instead of $\alpha_{3k_3}^*$ by some $k_3 \in \left\{\overline{1,\; M}\right\}$. So, this is the

$$\left\{\left\{\{\alpha_{il}^*\}_{i=1}^N\right\}_{l\in\left\{\overline{1,\; M}\right\}\setminus\{k_1,\; k_2,\; k_3\}} \cup \left\{\alpha_{1k_1}^{(0)},\; \{\alpha_{ik_1}^*\}_{i=2}^N\right\} \cup \left\{\alpha_{1k_2}^*,\; \alpha_{2k_2}^{(0)},\; \{\alpha_{ik_2}^*\}_{i=3}^N\right\} \cup\right.$$

$$\cup \left\{ \left\{ \left\{ \alpha^*_{ik_3} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{3k_3} \right\} \right\} \cup \left\{ \alpha^{(0)}_{3k_3} \right\} \right\} \right\}\text{-stack equilibrium.} \tag{55}$$

Thus, (55) means that

$$\sum_{l \in \left\{ \overline{1,\, M} \right\} \setminus \{k_1,\, k_2,\, k_3\}} K_1 \left(\alpha_{1l},\ \{\alpha^*_{il}\}_{i=2}^{N} \right) +$$
$$+ K_1 \left(\alpha_{1k_1},\ \left\{ \alpha^*_{ik_1} \right\}_{i=2}^{N} \right) + K_1 \left(\alpha_{1k_2},\ \alpha^{(0)}_{2k_2},\ \left\{ \alpha^*_{ik_2} \right\}_{i=3}^{N} \right) +$$
$$+ K_1 \left(\alpha_{1k_3},\ \left\{ \left\{ \alpha^*_{ik_3} \right\}_{i=2}^{N} \setminus \left\{ \alpha^*_{3k_3} \right\} \right\} \cup \left\{ \alpha^{(0)}_{3k_3} \right\} \right) \leqslant$$
$$\leqslant \sum_{l \in \left\{ \overline{1,\, M} \right\} \setminus \{k_1,\, k_2,\, k_3\}} K_1 \left(\{\alpha^*_{il}\}_{i=1}^{N} \right) +$$
$$+ K_1 \left(\alpha^{(0)}_{1k_1},\ \left\{ \alpha^*_{ik_1} \right\}_{i=2}^{N} \right) + K_1 \left(\left\{ \left\{ \alpha^*_{ik_2} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{2k_2} \right\} \right\} \cup \left\{ \alpha^{(0)}_{2k_2} \right\} \right) +$$
$$+ K_1 \left(\left\{ \left\{ \alpha^*_{ik_3} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{3k_3} \right\} \right\} \cup \left\{ \alpha^{(0)}_{3k_3} \right\} \right) \tag{56}$$

and

$$\sum_{l \in \left\{ \overline{1,\, M} \right\} \setminus \{k_1,\, k_2,\, k_3\}} K_2 \left(\left\{ \{\alpha^*_{il}\}_{i=1}^{N} \setminus \{\alpha^*_{2l}\} \right\} \cup \{\alpha_{2l}\} \right) +$$
$$+ K_2 \left(\alpha^{(0)}_{1k_1},\ \alpha_{2k_1},\ \left\{ \alpha^*_{ik_1} \right\}_{i=3}^{N} \right) +$$
$$+ K_2 \left(\left\{ \left\{ \alpha^*_{ik_2} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{2k_2} \right\} \right\} \cup \{\alpha_{2k_2}\} \right) + K_2 \left(\alpha^*_{1k_3},\ \alpha_{2k_3},\ \alpha^{(0)}_{3k_3},\ \left\{ \alpha^*_{ik_3} \right\}_{i=4}^{N} \right) \leqslant$$
$$\leqslant \sum_{l \in \left\{ \overline{1,\, M} \right\} \setminus \{k_1,\, k_2,\, k_3\}} K_2 \left(\{\alpha^*_{il}\}_{i=1}^{N} \right) +$$
$$+ K_2 \left(\alpha^{(0)}_{1k_1},\ \left\{ \alpha^*_{ik_1} \right\}_{i=2}^{N} \right) + K_2 \left(\left\{ \left\{ \alpha^*_{ik_2} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{2k_2} \right\} \right\} \cup \left\{ \alpha^{(0)}_{2k_2} \right\} \right) +$$
$$+ K_2 \left(\left\{ \left\{ \alpha^*_{ik_3} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{3k_3} \right\} \right\} \cup \left\{ \alpha^{(0)}_{3k_3} \right\} \right) \tag{57}$$

and

$$\sum_{l \in \left\{ \overline{1,\, M} \right\} \setminus \{k_1,\, k_2,\, k_3\}} K_3 \left(\left\{ \{\alpha^*_{il}\}_{i=1}^{N} \setminus \{\alpha^*_{3l}\} \right\} \cup \{\alpha_{3l}\} \right) +$$
$$+ K_3 \left(\alpha^{(0)}_{1k_1},\ \left\{ \left\{ \alpha^*_{ik_1} \right\}_{i=2}^{N} \setminus \left\{ \alpha^*_{3k_1} \right\} \right\} \cup \{\alpha_{3k_1}\} \right) +$$
$$+ K_3 \left(\alpha^*_{1k_2},\ \alpha^{(0)}_{2k_2},\ \alpha_{3k_2},\ \left\{ \alpha^*_{ik_2} \right\}_{i=4}^{N} \right) + K_3 \left(\left\{ \left\{ \alpha^*_{ik_3} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{3k_3} \right\} \right\} \cup \{\alpha_{3k_3}\} \right) \leqslant$$
$$\leqslant \sum_{l \in \left\{ \overline{1,\, M} \right\} \setminus \{k_1,\, k_2,\, k_3\}} K_3 \left(\{\alpha^*_{il}\}_{i=1}^{N} \right) +$$
$$+ K_3 \left(\alpha^{(0)}_{1k_1},\ \left\{ \alpha^*_{ik_1} \right\}_{i=2}^{N} \right) + K_3 \left(\left\{ \left\{ \alpha^*_{ik_2} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{2k_2} \right\} \right\} \cup \left\{ \alpha^{(0)}_{2k_2} \right\} \right) +$$

$$+K_3\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^N \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right) \tag{58}$$

and

$$\sum_{l\in\left\{\overline{1,\, M}\right\}\setminus\{k_1,\, k_2,\, k_3\}} K_n\left(\left\{\left\{\alpha^*_{il}\right\}_{i=1}^N \setminus \left\{\alpha^*_{nl}\right\}\right\} \cup \left\{\alpha_{nl}\right\}\right) +$$
$$+K_n\left(\alpha^{(0)}_{1k_1},\ \left\{\left\{\alpha^*_{ik_1}\right\}_{i=2}^N \setminus \left\{\alpha^*_{nk_1}\right\}\right\} \cup \left\{\alpha_{nk_1}\right\}\right) +$$
$$+K_n\left(\alpha^*_{1k_2},\ \alpha^{(0)}_{2k_2},\ \left\{\left\{\alpha^*_{ik_2}\right\}_{i=3}^N \setminus \left\{\alpha^*_{nk_2}\right\}\right\} \cup \left\{\alpha_{nk_2}\right\}\right) +$$
$$+K_n\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^N \setminus \left\{\alpha^*_{3k_3},\ \alpha^*_{nk_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3},\ \alpha_{nk_3}\right\}\right) \leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\, M}\right\}\setminus\{k_1,\, k_2,\, k_3\}} K_n\left(\left\{\alpha^*_{il}\right\}_{i=1}^N\right) + K_n\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^N\right) +$$
$$+K_n\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^N \setminus \left\{\alpha^*_{2k_2}\right\}\right\} \cup \left\{\alpha^{(0)}_{2k_2}\right\}\right) +$$
$$+K_n\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^N \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right) \quad \forall\, n = \overline{4,\ N}, \tag{59}$$

i. e., inequalities

$$K_1\left(\alpha_{1l},\ \left\{\alpha^*_{il}\right\}_{i=2}^N\right) \leqslant K_1\left(\left\{\alpha^*_{il}\right\}_{i=1}^N\right)$$
$$\forall\, \alpha_{1l} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right] \text{ and } \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2,\ k_3\} \tag{60}$$

and inequality

$$K_1\left(\alpha_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^N\right) + K_1\left(\alpha_{1k_2},\ \alpha^{(0)}_{2k_2},\ \left\{\alpha^*_{ik_2}\right\}_{i=3}^N\right) +$$
$$+K_1\left(\alpha_{1k_3},\ \left\{\left\{\alpha^*_{ik_3}\right\}_{i=2}^N \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right) \leqslant$$
$$\leqslant K_1\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=2}^N\right) + K_1\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^N \setminus \left\{\alpha^*_{2k_2}\right\}\right\} \cup \left\{\alpha^{(0)}_{2k_2}\right\}\right) +$$
$$+K_1\left(\left\{\left\{\alpha^*_{ik_3}\right\}_{i=1}^N \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right) \forall\, \alpha_{1k_1} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]$$
$$\text{and } \forall\, \alpha_{1k_2} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right] \text{ and } \forall\, \alpha_{1k_3} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right] \tag{61}$$

hold along with (23) for $n = 1$, inequalities

$$K_2\left(\left\{\left\{\alpha^*_{il}\right\}_{i=1}^N \setminus \left\{\alpha^*_{2l}\right\}\right\} \cup \left\{\alpha_{2l}\right\}\right) \leqslant K_2\left(\left\{\alpha^*_{il}\right\}_{i=1}^N\right)$$
$$\forall\, \alpha_{2l} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right] \text{ and } \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \{k_1,\ k_2,\ k_3\} \tag{62}$$

and inequality

$$K_2\left(\alpha^{(0)}_{1k_1},\ \alpha_{2k_1},\ \left\{\alpha^*_{ik_1}\right\}_{i=3}^N\right) + K_2\left(\left\{\left\{\alpha^*_{ik_2}\right\}_{i=1}^N \setminus \left\{\alpha^*_{2k_2}\right\}\right\} \cup \left\{\alpha_{2k_2}\right\}\right) +$$

$$+K_2\left(\alpha^*_{1k_3},\ \alpha_{2k_3},\ \alpha^{(0)}_{3k_3},\ \left\{\alpha^*_{ik_3}\right\}^N_{i=4}\right) \leqslant$$
$$\leqslant K_2\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}^N_{i=2}\right) + K_2\left(\left\{\left\{\alpha^*_{ik_2}\right\}^N_{i=1} \setminus \left\{\alpha^*_{2k_2}\right\}\right\} \cup \left\{\alpha^{(0)}_{2k_2}\right\}\right) +$$
$$+K_2\left(\left\{\left\{\alpha^*_{ik_3}\right\}^N_{i=1} \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right) \forall\, \alpha_{2k_1} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right]$$
$$\text{and } \forall\, \alpha_{2k_2} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right] \text{ and } \forall\, \alpha_{2k_3} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right] \tag{63}$$

hold along with (23) for $n = 2$, inequalities

$$K_3\left(\left\{\left\{\alpha^*_{il}\right\}^N_{i=1} \setminus \left\{\alpha^*_{3l}\right\}\right\} \cup \left\{\alpha_{3l}\right\}\right) \leqslant K_3\left(\left\{\alpha^*_{il}\right\}^N_{i=1}\right)$$
$$\forall\, \alpha_{3l} \in \left[x_3^{(\min)};\ x_3^{(\max)}\right] \text{ and } \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \left\{k_1,\ k_2,\ k_3\right\} \tag{64}$$

and inequality

$$K_3\left(\alpha^{(0)}_{1k_1},\ \left\{\left\{\alpha^*_{ik_1}\right\}^N_{i=2} \setminus \left\{\alpha^*_{3k_1}\right\}\right\} \cup \left\{\alpha_{3k_1}\right\}\right) +$$
$$+K_3\left(\alpha^*_{1k_2},\ \alpha^{(0)}_{2k_2},\ \alpha_{3k_2},\ \left\{\alpha^*_{ik_2}\right\}^N_{i=4}\right) +$$
$$+K_3\left(\left\{\left\{\alpha^*_{ik_3}\right\}^N_{i=1} \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha_{3k_3}\right\}\right) \leqslant$$
$$\leqslant K_3\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}^N_{i=2}\right) + K_3\left(\left\{\left\{\alpha^*_{ik_2}\right\}^N_{i=1} \setminus \left\{\alpha^*_{2k_2}\right\}\right\} \cup \left\{\alpha^{(0)}_{2k_2}\right\}\right) +$$
$$+K_3\left(\left\{\left\{\alpha^*_{ik_3}\right\}^N_{i=1} \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right) \forall\, \alpha_{3k_1} \in \left[x_3^{(\min)};\ x_3^{(\max)}\right]$$
$$\text{and } \forall\, \alpha_{3k_2} \in \left[x_3^{(\min)};\ x_3^{(\max)}\right] \text{ and } \forall\, \alpha_{3k_3} \in \left[x_3^{(\min)};\ x_3^{(\max)}\right] \tag{65}$$

hold along with (23) for $n = 3$, inequalities

$$K_n\left(\left\{\left\{\alpha^*_{il}\right\}^N_{i=1} \setminus \left\{\alpha^*_{nl}\right\}\right\} \cup \left\{\alpha_{nl}\right\}\right) \leqslant K_n\left(\left\{\alpha^*_{il}\right\}^N_{i=1}\right)$$
$$\forall\, \alpha_{nl} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \text{ and } \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \left\{k_1,\ k_2,\ k_3\right\} \text{ and } \forall\, n = \overline{4,\ N} \tag{66}$$

and inequality

$$K_n\left(\alpha^{(0)}_{1k_1},\ \left\{\left\{\alpha^*_{ik_1}\right\}^N_{i=2} \setminus \left\{\alpha^*_{nk_1}\right\}\right\} \cup \left\{\alpha_{nk_1}\right\}\right) +$$
$$+K_n\left(\alpha^*_{1k_2},\ \alpha^{(0)}_{2k_2},\ \left\{\left\{\alpha^*_{ik_2}\right\}^N_{i=3} \setminus \left\{\alpha^*_{nk_2}\right\}\right\} \cup \left\{\alpha_{nk_2}\right\}\right) +$$
$$+K_n\left(\left\{\left\{\alpha^*_{ik_3}\right\}^N_{i=1} \setminus \left\{\alpha^*_{3k_3},\ \alpha^*_{nk_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3},\ \alpha_{nk_3}\right\}\right) \leqslant$$
$$\leqslant K_n\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}^N_{i=2}\right) + K_n\left(\left\{\left\{\alpha^*_{ik_2}\right\}^N_{i=1} \setminus \left\{\alpha^*_{2k_2}\right\}\right\} \cup \left\{\alpha^{(0)}_{2k_2}\right\}\right) +$$
$$+K_n\left(\left\{\left\{\alpha^*_{ik_3}\right\}^N_{i=1} \setminus \left\{\alpha^*_{3k_3}\right\}\right\} \cup \left\{\alpha^{(0)}_{3k_3}\right\}\right) \ \forall\, n = \overline{4,\ N} \tag{67}$$

hold along with (23). Plugging $\alpha_{1k_2} = \alpha^*_{1k_2}$ and $\alpha_{1k_3} = \alpha^*_{1k_3}$ in the left side of inequality (61) gives inequality

$$K_1\left(\alpha_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}^N_{i=2}\right) \leqslant K_1\left(\alpha^{(0)}_{1k_1},\ \left\{\alpha^*_{ik_1}\right\}^N_{i=2}\right) \ \forall\, \alpha_{1k_1} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right], \tag{68}$$

plugging $\alpha_{2k_2} = \alpha_{2k_2}^{(0)}$ and $\alpha_{2k_3} = \alpha_{2k_3}^{*}$ in the left side of inequality (63) gives inequality

$$K_2\left(\alpha_{1k_1}^{(0)},\ \alpha_{2k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=3}^{N}\right) \leqslant K_2\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)$$
$$\forall\, \alpha_{2k_1} \in \left[x_2^{(\min)};\ x_2^{(\max)}\right], \tag{69}$$

plugging $\alpha_{3k_2} = \alpha_{3k_2}^{*}$ and $\alpha_{3k_3} = \alpha_{3k_3}^{(0)}$ in the left side of inequality (65) gives inequality

$$K_3\left(\alpha_{1k_1}^{(0)},\ \left\{\left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N} \setminus \left\{\alpha_{3k_1}^{*}\right\}\right\} \cup \left\{\alpha_{3k_1}\right\}\right) \leqslant K_3\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)$$
$$\forall\, \alpha_{3k_1} \in \left[x_3^{(\min)};\ x_3^{(\max)}\right], \tag{70}$$

and plugging $\alpha_{nk_2} = \alpha_{nk_2}^{*}$ and $\alpha_{nk_3} = \alpha_{nk_3}^{*}$ in the left side of inequality (67) for $n = \overline{4,\ N}$ gives inequality

$$K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N} \setminus \left\{\alpha_{nk_1}^{*}\right\}\right\} \cup \left\{\alpha_{nk_1}\right\}\right) \leqslant K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)$$
$$\forall\, \alpha_{nk_1} \in \left[x_n^{(\min)};\ x_n^{(\max)}\right] \text{ and } \forall\, n = \overline{4,\ N}. \tag{71}$$

Inequalities (68) — (71) imply that $\left\{\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right\}$ is a pure-strategy equilibrium, which is impossible. Therefore, (55) is false. The same conclusion is valid for a three-person non-cooperative game, where (57), (58), (63), (65) are written by retaining $\left\{\alpha_{ik_3}^{*}\right\}_{i=4}^{N} = \emptyset$, $\left\{\alpha_{ik_2}^{*}\right\}_{i=4}^{N} = \emptyset$, and (59), (66), (67), (71) are omitted. The impossibility of the other pure-strategy equilibrium for the remaining players' subsets in the case of three different strategies at three players is proved symmetrically.

Finally, suppose that the other pure-strategy equilibrium differs from (27) in that the first player uses some $\alpha_{1k_1}^{(0)} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]$ instead of $\alpha_{1k_1}^{*}$ by some $k_1 \in \left\{\overline{1,\ M}\right\}$ and some $\alpha_{1k_2}^{(0)} \in \left[x_1^{(\min)};\ x_1^{(\max)}\right]$ instead of $\alpha_{1k_2}^{*}$ by some $k_2 \in \left\{\overline{1,\ M}\right\}$. The respective

$$\left\{\left\{\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right\}_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}} \cup\right.$$
$$\left.\cup \left\{\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right\} \cup \left\{\alpha_{1k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\right\}\right\}\text{-stack equilibrium} \tag{72}$$

means that

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}} K_1\left(\alpha_{1l},\ \left\{\alpha_{il}^{*}\right\}_{i=2}^{N}\right) +$$
$$+K_1\left(\alpha_{1k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right) + K_1\left(\alpha_{1k_2},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\right) \leqslant$$
$$\leqslant \sum_{l\in\left\{\overline{1,\ M}\right\}\setminus\{k_1,\ k_2\}} K_1\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right) +$$

$$+K_1\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_1\left(\alpha_{1k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\right) \tag{73}$$

and

$$\sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1,\ k_2\}} K_n\left(\left\{\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\backslash\left\{\alpha_{nl}^{*}\right\}\right\}\cup\left\{\alpha_{nl}\right\}\right)+$$
$$+K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\backslash\left\{\alpha_{nk_1}^{*}\right\}\right\}\cup\left\{\alpha_{nk_1}\right\}\right)+$$
$$+K_n\left(\alpha_{1k_2}^{(0)},\ \left\{\left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\backslash\left\{\alpha_{nk_2}^{*}\right\}\right\}\cup\left\{\alpha_{nk_2}\right\}\right)\leqslant$$
$$\leqslant\sum_{l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1,\ k_2\}} K_n\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right)+$$
$$+K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_n\left(\alpha_{1k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\right)\ \ \forall\, n=\overline{2,\ N}, \tag{74}$$

i. e., inequalities (46) and inequality

$$K_1\left(\alpha_{1k_1},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_1\left(\alpha_{1k_2},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\right)\leqslant$$
$$\leqslant K_1\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_1\left(\alpha_{1k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\right)$$
$$\forall\ \alpha_{1k_1}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right]\ \text{ and }\ \forall\ \alpha_{1k_2}\in\left[x_1^{(\min)};\ x_1^{(\max)}\right] \tag{75}$$

hold along with (23) for $n=1$, inequalities

$$K_n\left(\left\{\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\backslash\left\{\alpha_{nl}^{*}\right\}\right\}\cup\left\{\alpha_{nl}\right\}\right)\leqslant K_n\left(\left\{\alpha_{il}^{*}\right\}_{i=1}^{N}\right)$$
$$\forall\ \alpha_{nl}\in\left[x_n^{(\min)};\ x_n^{(\max)}\right]\ \text{ and }\ \forall\, l\in\left\{\overline{1,\ M}\right\}\backslash\{k_1,\ k_2\}\ \text{ and }\ \forall\, n=\overline{2,\ N} \tag{76}$$

and inequality

$$K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\backslash\left\{\alpha_{nk_1}^{*}\right\}\right\}\cup\left\{\alpha_{nk_1}\right\}\right)+$$
$$+K_n\left(\alpha_{1k_2}^{(0)},\ \left\{\left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\backslash\left\{\alpha_{nk_2}^{*}\right\}\right\}\cup\left\{\alpha_{nk_2}\right\}\right)\leqslant$$
$$\leqslant K_n\left(\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right)+K_n\left(\alpha_{1k_2}^{(0)},\ \left\{\alpha_{ik_2}^{*}\right\}_{i=2}^{N}\right)\ \ \forall\, n=\overline{2,\ N} \tag{77}$$

hold along with (23). Plugging $\alpha_{1k_2}=\alpha_{1k_2}^{(0)}$ in the left side of inequality (75) and plugging $\alpha_{nk_2}=\alpha_{nk_2}^{*}$ in the left side of inequality (77) for $n=\overline{2,\ N}$ gives inequalities (68) — (71), which are impossible due to $\left\{\alpha_{1k_1}^{(0)},\ \left\{\alpha_{ik_1}^{*}\right\}_{i=2}^{N}\right\}$ is not a pure-strategy equilibrium. So, the supposition about (72) is contradictory. The same conclusion is valid for a two-person non-cooperative game, where (70), (71) are omitted, and it is valid for a three-person non-cooperative game, where (71) is omitted. The impossibility of the other pure-strategy equilibrium for the remaining players in such a case (of two intervals) is proved symmetrically. The impossibility of other pure-strategy equilibria differing from (27) in that the players use some other values at intervals is proved symmetrically as well. □

Therefore, Theorem 3 along with Theorem 2 allows obtaining the single pure-strategy solution of game (6) directly from equilibria in games (13). The application of these assertions significantly simplifies the solving of game (6). Under conditions of the assertions, game (6) is "discretized" or "broken" into simpler N-person games, whereupon their equilibria are stacked [18, 19].

But what if the conditions are inverted? Does the equilibrium singularity in games (13) change when the single pure-strategy equilibrium of game (6) is already known? This question is answered by the following assertion.

Theorem 4. *If game (6) on product (7) by conditions (1) — (5) and (8) — (12) has a single equilibrium situation in pure strategies, then each of M games (13) by (8) — (12) and (15) — (18) has a single pure-strategy equilibrium, which is the respective interval part of the game (6) equilibrium.*

Proof. Let game (6) have (27) which is single. This implies that inequalities (26) hold. Plugging

$$\alpha_{nl} = \alpha^*_{nl} \ \ \forall\, l \in \left\{\overline{1,\ M}\right\} \setminus \{k_*\}$$

in the left side of inequalities (26) gives inequalities

$$K_n \left(\left\{ \left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{nk_*} \right\} \right\} \cup \left\{ \alpha_{nk_*} \right\} \right) \leqslant K_n \left(\left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \right)$$
$$\forall\, \alpha_{nk_*} \in \left[x_n^{(\min)};\ x_n^{(\max)} \right] \ \text{ and } \ \forall\, n = \overline{1,\ N} \tag{78}$$

whence $\left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N}$ is a pure-strategy equilibrium at the k_*-th interval (in the k_*-th game) for every $k_* \in \left\{\overline{1,\ M}\right\}$.

Suppose that $\exists\ k_0 \in \left\{\overline{1,\ M}\right\}$ such that $\left\{ \alpha^{(0)}_{1k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=2}^{N} \right\}$ is an equilibrium by $\alpha^{(0)}_{1k_0} \neq \alpha^*_{1k_0}$. Then inequalities

$$K_1 \left(\alpha_{1k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=2}^{N} \right) \leqslant K_1 \left(\alpha^{(0)}_{1k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=2}^{N} \right) \ \ \forall\, \alpha_{1k_0} \in \left[x_1^{(\min)};\ x_1^{(\max)} \right] \tag{79}$$

and

$$K_n \left(\left\{ \left\{ \alpha^{(0)}_{1k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=2}^{N} \right\} \setminus \left\{ \alpha^*_{nk_0} \right\} \right\} \cup \left\{ \alpha_{nk_0} \right\} \right) \leqslant K_n \left(\alpha^{(0)}_{1k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=2}^{N} \right)$$
$$\forall\, \alpha_{nk_0} \in \left[x_n^{(\min)};\ x_n^{(\max)} \right] \ \text{ and } \ \forall\, n = \overline{2,\ N} \tag{80}$$

hold, whence inequalities

$$\sum_{k_* \in \left\{\overline{1,\ M}\right\} \setminus \{k_0\}} K_1 \left(\left\{ \left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{1k_*} \right\} \right\} \cup \left\{ \alpha_{1k_*} \right\} \right) + K_1 \left(\alpha_{1k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=2}^{N} \right) \leqslant$$
$$\leqslant \sum_{k_* \in \left\{\overline{1,\ M}\right\} \setminus \{k_0\}} K_1 \left(\left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \right) + K_1 \left(\alpha^{(0)}_{1k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=2}^{N} \right) \tag{81}$$

and

$$\sum_{k_* \in \left\{\overline{1,\ M}\right\} \backslash \{k_0\}} K_n \left(\left\{ \left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \backslash \left\{ \alpha^*_{nk_*} \right\} \right\} \cup \left\{ \alpha_{nk_*} \right\} \right) +$$

$$+ K_n \left(\left\{ \left\{ \alpha^{(0)}_{1k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=2}^{N} \right\} \backslash \left\{ \alpha^*_{nk_0} \right\} \right\} \cup \left\{ \alpha_{nk_0} \right\} \right) \leqslant$$

$$\leqslant \sum_{k_* \in \left\{\overline{1,\ M}\right\} \backslash \{k_0\}} K_n \left(\left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \right) + K_n \left(\alpha^{(0)}_{1k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=2}^{N} \right) \quad \forall\, n = \overline{2,\ N} \tag{82}$$

must hold as well. However, inequalities (81) and (82) imply that there is the

$$\left\{ \left\{ \left\{ \alpha^*_{il} \right\}_{i=1}^{N} \right\}_{l \in \left\{\overline{1,\ M}\right\} \backslash \{k_0\}} \cup \left\{ \alpha^{(0)}_{1k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=2}^{N} \right\} \right\}\text{-stack equilibrium,}$$

which is impossible. Supposing that $\left\{ \alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right\}$ is an equilibrium by $\alpha^{(0)}_{1k_0} \neq \alpha^*_{1k_0}$ and $\alpha^{(0)}_{2k_0} \neq \alpha^*_{2k_0}$ leads to inequalities

$$K_1 \left(\alpha_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right) \leqslant K_1 \left(\alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right)$$

$$\forall\, \alpha_{1k_0} \in \left[x_1^{(\min)};\ x_1^{(\max)} \right] \tag{83}$$

and

$$K_2 \left(\alpha^{(0)}_{1k_0},\ \alpha_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right) \leqslant K_2 \left(\alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right)$$

$$\forall\, \alpha_{2k_0} \in \left[x_2^{(\min)};\ x_2^{(\max)} \right] \tag{84}$$

and

$$K_n \left(\left\{ \left\{ \alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right\} \backslash \left\{ \alpha^*_{nk_0} \right\} \right\} \cup \left\{ \alpha_{nk_0} \right\} \right) \leqslant$$

$$\leqslant K_n \left(\alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right)$$

$$\forall\, \alpha_{nk_0} \in \left[x_n^{(\min)};\ x_n^{(\max)} \right] \ \text{ and } \ \forall\, n = \overline{3,\ N}. \tag{85}$$

Inequalities (83) — (85) imply that inequalities

$$\sum_{k_* \in \left\{\overline{1,\ M}\right\} \backslash \{k_0\}} K_1 \left(\left\{ \left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \backslash \left\{ \alpha^*_{1k_*} \right\} \right\} \cup \left\{ \alpha_{1k_*} \right\} \right) +$$

$$+ K_1 \left(\alpha_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right) \leqslant$$

$$\leqslant \sum_{k_* \in \left\{\overline{1,\ M}\right\} \backslash \{k_0\}} K_1 \left(\left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \right) + K_1 \left(\alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right) \tag{86}$$

and

$$\sum_{k_* \in \left\{\overline{1,\, M}\right\} \backslash \{k_0\}} K_2 \left(\left\{ \left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{2k_*} \right\} \right\} \cup \left\{ \alpha_{2k_*} \right\} \right) +$$

$$+ K_2 \left(\alpha^{(0)}_{1k_0},\ \alpha_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right) \leqslant$$

$$\leqslant \sum_{k_* \in \left\{\overline{1,\, M}\right\} \backslash \{k_0\}} K_2 \left(\left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \right) + K_2 \left(\alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right) \tag{87}$$

and

$$\sum_{k_* \in \left\{\overline{1,\, M}\right\} \backslash \{k_0\}} K_n \left(\left\{ \left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \setminus \left\{ \alpha^*_{nk_*} \right\} \right\} \cup \left\{ \alpha_{nk_*} \right\} \right) +$$

$$+ K_n \left(\left\{ \left\{ \alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right\} \setminus \left\{ \alpha^*_{nk_0} \right\} \right\} \cup \left\{ \alpha_{nk_0} \right\} \right) \leqslant$$

$$\leqslant \sum_{k_* \in \left\{\overline{1,\, M}\right\} \backslash \{k_0\}} K_n \left(\left\{ \alpha^*_{ik_*} \right\}_{i=1}^{N} \right) + K_n \left(\alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right)$$

$$\forall\, n = \overline{3,\ N} \tag{88}$$

must hold as well. Then inequalities (83) — (85) imply that there is the

$$\left\{ \left\{ \left\{ \alpha^*_{il} \right\}_{i=1}^{N} \right\}_{l \in \left\{\overline{1,\, M}\right\} \backslash \{k_0\}} \cup \left\{ \alpha^{(0)}_{1k_0},\ \alpha^{(0)}_{2k_0},\ \left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} \right\} \right\}\text{-stack equilibrium,}$$

which is impossible again. The same conclusion is valid for a two-person non-cooperative game, where (83), (84), (86), (87) are written by retaining $\left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} = \emptyset$, $\left\{ \alpha^*_{ik_0} \right\}_{i=3}^{N} = \emptyset$, and (85), (88) are omitted. The impossibility of other pure-strategy equilibrium cases in "short" games (13) is proved symmetrically. □

In finite games of three players and more, which are a partial case of non-cooperative games, the case when every "short" game has just a single pure-strategy equilibrium seems to be rarer than the case with multiple equilibria. Obviously, the equilibrium singleness likelihood expectedly decays as the number of players increases. This, however, does not diminish the importance of Theorem 2 along with Theorem 3 and Theorem 4. These assertions allow to build a simpler proof of a more generalized assertion.

Theorem 5. *If each of M games (13) by (8) — (12) and (15) — (18) has a nonempty set of equilibrium situations in pure strategies, and game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of these games, then every pure-strategy equilibrium in game (6) is a stack of any respective M equilibria in games (13). Apart from the stack, there are no other pure-strategy equilibria in game (6).*

Proof. Let the l-th game have J_l equilibria $\left\{ \left\{ \alpha^*_{ilj_l} \right\}_{i=1}^{N} \right\}_{j_l=1}^{J_l}$ by $J_l \in \mathbb{N}$, where

$$\alpha^*_{nlj_l} \in \left[x_n^{(\min)};\ x_n^{(\max)} \right] \quad \forall\, n = \overline{1,\ N}.$$

Then

$$K_n\left(\left\{\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{nlj_l}\right\}\right\}\cup\{\alpha_{nl}\}\right)\leqslant K_n\left(\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\right)$$
$$\forall\,\alpha_{nl}\in\left[x_n^{(\min)};\ x_n^{(\max)}\right]\ \text{ and }\ \forall\,n=\overline{1,\ N} \tag{89}$$

whence

$$\sum_{l=1}^{M}K_n\left(\left\{\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\setminus\left\{\alpha^*_{nlj_l}\right\}\right\}\cup\{\alpha_{nl}\}\right)\leqslant\sum_{l=1}^{M}K_n\left(\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\right)\ \ \forall\,n=\overline{1,\ N}. \tag{90}$$

Inequalities (90) directly imply the

$$\left\{\left\{\alpha^*_{ilj_l}\right\}_{i=1}^{N}\right\}_{l=1}^{M}\text{-stack equilibrium} \tag{91}$$

for every $j_l\in\left\{\overline{1,\ J_l}\right\}$ by $l=\overline{1,\ M}$. Apart from stacks (91), there are no other pure-strategy equilibria in game (6) owing to Theorem 4 along with Theorem 3. □

It is quite obvious that Theorems 2 — 5 are valid for any non-cooperative games whose players are constrained (forced) to use staircase-function strategies, i. e., they are valid for finite non-cooperative games (with staircase-function strategies) as well. It remains only to study a possibility of equilibria in mixed strategies in such finite games.

5. Representation by a succession of finite games

Along with discrete time intervals, players may be forced to act within a finite subset of possible values of their pure strategies. That is, these values are

$$x_n^{(\min)}=x_n^{(0)}<x_n^{(1)}<x_n^{(2)}<\ldots<x_n^{(Q_n-1)}<x_n^{(Q_n)}=x_n^{(\max)} \tag{92}$$

for the n-th player, $Q_n\in\mathbb{N}\ \forall\,n=\overline{1,\ N}$. Then the succession of M continuous games (13) by (8) — (12) and (15) — (18) becomes a succession of M finite games

$$\left\langle\left\{\left\{x_i^{(m_i-1)}\right\}_{m_i=1}^{Q_i+1}\right\}_{i=1}^{N},\ \{\mathbf{H}_{il}\}_{i=1}^{N}\right\rangle \tag{93}$$

with the n-th player's payoff matrix

$$\mathbf{H}_{nl}=\left[h_{nl\boldsymbol{\Omega}}\right]_{\mathscr{F}} \tag{94}$$

whose format is

$$\mathscr{F}=\mathop{\times}_{n=1}^{N}\left(Q_n+1\right) \tag{95}$$

and elements are

$$h_{nl\mathbf{\Omega}} = \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right) d\mu(t) \ \text{ for } \ l = \overline{1,\ M-1} \tag{96}$$

and

$$h_{nM\mathbf{\Omega}} = \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right) d\mu(t) \tag{97}$$

by indexing

$$\mathbf{\Omega} = \{\omega_k\}_{k=1}^{N},\ \ \omega_k \in \left\{\overline{1,\ Q_k+1}\right\} \ \ \forall\, k = \overline{1,\ N}. \tag{98}$$

It is well-known that a finite non-cooperative game always has an equilibrium either in pure or mixed strategies. So, if game (6) is made equivalent to a series of finite games (or, in other words, is represented by a succession of finite games), then it is easy to see that, unlike the representation with continuous games (13) by (8) — (12) and (15) — (18), the game always has a solution (at least, in mixed strategies).

Theorem 6. *If game (6) on product (7) by conditions (1) — (5) is equivalent to the succession of M finite games (93) by (94) — (98), then the game is always solved as a stack of respective equilibria in these finite games. Apart from the stack, there are no other equilibria in game (6).*

Proof. An equilibrium situation in the finite game always exists, either in pure or mixed strategies. Denote by

$$\mathbf{U}_{nl} = \left[u_{nl}^{(m_n)}\right]_{1\times(Q_n+1)}$$

a mixed strategy of the n-th player in finite game (93). The respective set of mixed strategies of this player is

$$\mathcal{U}_n = \left\{\mathbf{U}_{nl} \in \mathbb{R}^{Q_n+1} : u_{nl}^{(m_n)} \geqslant 0,\ \sum_{m_n=1}^{Q_n+1} u_{nl}^{(m_n)} = 1\right\}, \tag{99}$$

so $\mathbf{U}_{nl} \in \mathcal{U}_n$, and $\{\mathbf{U}_{il}\}_{i=1}^{N}$ is a situation in game (93), where J_l equilibria exist, $J_l \in \mathbb{N}$. Let $\left\{\left\{\mathbf{U}_{ilj_l}^{*}\right\}_{i=1}^{N}\right\}_{l=1}^{M}$ be equilibria in M games (93) by (94) — (98), where

$$\mathbf{U}_{nlj_l}^{*} = \left[u_{nlj_l}^{(m_n)*}\right]_{1\times(Q_n+1)} \in \mathcal{U}_n. \tag{100}$$

Henceforward, the proof is similar to that in Theorem 5. For equilibria $\left\{\left\{\mathbf{U}_{ilj_l}^{*}\right\}_{i=1}^{N}\right\}_{l=1}^{M}$ by (100), inequalities

$$\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}} \left(h_{nl\mathbf{\Omega}} u_{nl}^{(m_n)} \prod_{\substack{k=\overline{1,\ N}\\ k\neq n}} u_{klj_l}^{(m_k)*}\right) =$$

$$= \sum_{\substack{m_k=\overline{1,\ Q_k+1} \\ k=\overline{1,\ N}}} \left(\left(u_{nl}^{(m_n)} \prod_{\substack{k=\overline{1,\ N} \\ k \neq n}} u_{klj_l}^{(m_k)*} \right) \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n \left(\left\{ x_i^{(m_i-1)} \right\}_{i=1}^{N},\ t \right) d\mu(t) \right) \leqslant$$

$$\leqslant \sum_{\substack{m_k=\overline{1,\ Q_k+1} \\ k=\overline{1,\ N}}} \left(\left(\prod_{k=\overline{1,\ N}} u_{klj_l}^{(m_k)*} \right) \int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)} f_n \left(\left\{ x_i^{(m_i-1)} \right\}_{i=1}^{N},\ t \right) d\mu(t) \right) =$$

$$= \sum_{\substack{m_k=\overline{1,\ Q_k+1} \\ k=\overline{1,\ N}}} \left(h_{nl\mathbf{\Omega}} \prod_{k=\overline{1,\ N}} u_{klj_l}^{(m_k)*} \right)$$

$$\forall\, \mathbf{U}_{nl} = \left[u_{nl}^{(m_n)} \right]_{1 \times (Q_n+1)} \in \mathcal{U}_n \ \text{ for } \ l = \overline{1,\ M-1} \ \ \forall\, n = \overline{1,\ N}, \tag{101}$$

$$\sum_{\substack{m_k=\overline{1,\ Q_k+1} \\ k=\overline{1,\ N}}} \left(h_{nM\mathbf{\Omega}} u_{nM}^{(m_n)} \prod_{\substack{k=\overline{1,\ N} \\ k \neq n}} u_{kMj_M}^{(m_k)*} \right) =$$

$$= \sum_{\substack{m_k=\overline{1,\ Q_k+1} \\ k=\overline{1,\ N}}} \left(\left(u_{nM}^{(m_n)} \prod_{\substack{k=\overline{1,\ N} \\ k \neq n}} u_{kMj_M}^{(m_k)*} \right) \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n \left(\left\{ x_i^{(m_i-1)} \right\}_{i=1}^{N},\ t \right) d\mu(t) \right) \leqslant$$

$$\leqslant \sum_{\substack{m_k=\overline{1,\ Q_k+1} \\ k=\overline{1,\ N}}} \left(\left(\prod_{k=\overline{1,\ N}} u_{kMj_M}^{(m_k)*} \right) \int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]} f_n \left(\left\{ x_i^{(m_i-1)} \right\}_{i=1}^{N},\ t \right) d\mu(t) \right) =$$

$$= \sum_{\substack{m_k=\overline{1,\ Q_k+1} \\ k=\overline{1,\ N}}} \left(h_{nM\mathbf{\Omega}} \prod_{k=\overline{1,\ N}} u_{kMj_M}^{(m_k)*} \right)$$

$$\forall\, \mathbf{U}_{nM} = \left[u_{nM}^{(m_n)} \right]_{1 \times (Q_n+1)} \in \mathcal{U}_n \ \text{ and } \ \forall\, n = \overline{1,\ N} \tag{102}$$

hold. So, inequalities

$$\sum_{l=1}^{M-1} \sum_{\substack{m_k=\overline{1,\ Q_k+1} \\ k=\overline{1,\ N}}} \left(h_{nl\mathbf{\Omega}} u_{nl}^{(m_n)} \prod_{\substack{k=\overline{1,\ N} \\ k \neq n}} u_{klj_l}^{(m_k)*} \right) +$$

$$+\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(h_{nM\boldsymbol{\Omega}}u_{nM}^{(m_n)}\prod_{\substack{k=\overline{1,\ N}\\ k\neq n}}u_{kMj_M}^{(m_k)*}\right)=$$

$$=\sum_{l=1}^{M-1}\left(\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(\left(u_{nl}^{(m_n)}\prod_{\substack{k=\overline{1,\ N}\\ k\neq n}}u_{klj_l}^{(m_k)*}\right)\int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)}f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right)d\mu(t)\right)\right)+$$

$$+\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(\left(u_{nM}^{(m_n)}\prod_{\substack{k=\overline{1,\ N}\\ k\neq n}}u_{kMj_M}^{(m_k)*}\right)\int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]}f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right)d\mu(t)\right)\leqslant$$

$$\leqslant\sum_{l=1}^{M-1}\left(\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(\left(\prod_{k=\overline{1,\ N}}u_{klj_l}^{(m_k)*}\right)\int\limits_{\left[\tau^{(l-1)};\ \tau^{(l)}\right)}f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right)d\mu(t)\right)\right)+$$

$$+\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(\left(\prod_{k=\overline{1,\ N}}u_{kMj_M}^{(m_k)*}\right)\int\limits_{\left[\tau^{(M-1)};\ \tau^{(M)}\right]}f_n\left(\left\{x_i^{(m_i-1)}\right\}_{i=1}^{N},\ t\right)d\mu(t)\right)=$$

$$=\sum_{l=1}^{M-1}\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(h_{nl\boldsymbol{\Omega}}\prod_{k=\overline{1,\ N}}u_{klj_l}^{(m_k)*}\right)+$$

$$+\sum_{\substack{m_k=\overline{1,\ Q_k+1}\\ k=\overline{1,\ N}}}\left(h_{nM\boldsymbol{\Omega}}\prod_{k=\overline{1,\ N}}u_{kMj_M}^{(m_k)*}\right)\ \forall\, n=\overline{1,\ N} \tag{103}$$

hold as well. Therefore, the stack of successive equilibria $\left\{\left\{\mathbf{U}_{ilj_l}^{*}\right\}_{i=1}^{N}\right\}_{l=1}^{M}$ is an equilibrium in game (6). The sub-assertion of that, apart from such stacks, there are no other equilibria in game (6) is proved similarly to Theorem 4 along with Theorem 3. □

Clearly, inequalities (89) by $l=\overline{1,\ M}$ are a partial case of inequalities (101), (102). Inequalities (90) are a partial case of inequalities (103). In a way, Theorem 6 is a generalization of Theorem 5 for the case of finite game (6), which is correspondingly defined on a product of staircase-function finite spaces. Nevertheless, stacking up pure-strategy equilibria and mixed-strategy equilibria of $\underset{n=1}{\overset{N}{\times}}(Q_n+1)$ finite games (93) can be cumbersome. The best case is when every "short" game has a single pure-strategy equilibrium, although the likelihood of the best case is low.

The likeliest case is when those M finite games have multiple pure-strategy equilibria and mixed-strategy equilibria. To hit on a series of single-pure-strategy-equilibrium finite games, plainly speaking, many tries should be done. For instance, $5\times5\times5\times5$ games, in which payoffs are generated by a $5\times5\times5\times5$ standard-normally-distributed array multiplied by 10 and rounded to the nearest integers towards $-\infty$, have roughly 27.5 % mixed-strategy equilibria only. The percentage rate of the case when the game has one pure-strategy equilibrium is at least 36 %. Meanwhile, these rates for $5\times5\times5$ games are 28 % and 37 %, respectively.

6. An example of solving a finite game

To exemplify how the suggested method solves finite games defined on a product of staircase-function spaces (which are obviously finite), consider a case in which $t \in [0;\ 0.16\pi]$, the set of pure strategies of the first player is

$$X_1 = \{x_1(t),\ t \in [0;\ 0.16\pi]:\ 2 \leqslant x_1(t) \leqslant 3\} \subset \mathbb{L}_2[0;\ 0.16\pi], \tag{104}$$

the set of pure strategies of the second player is

$$X_2 = \{x_2(t),\ t \in [0;\ 0.16\pi]:\ 4 \leqslant x_2(t) \leqslant 4.75\} \subset \mathbb{L}_2[0;\ 0.16\pi], \tag{105}$$

and the set of pure strategies of the third player is

$$X_3 = \{x_3(t),\ t \in [0;\ 0.16\pi]:\ 1 \leqslant x_3(t) \leqslant 1.5\} \subset \mathbb{L}_2[0;\ 0.16\pi], \tag{106}$$

and the set of pure strategies of the fourth player is

$$X_4 = \{x_4(t),\ t \in [0;\ 0.16\pi]:\ 3 \leqslant x_4(t) \leqslant 3.4\} \subset \mathbb{L}_2[0;\ 0.16\pi]. \tag{107}$$

The players' payoff functionals (4) are

$$K_1(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)) = \\ = \int\limits_{[0;\ 0.16\pi]} \sin(0.2x_1x_2x_3x_4t)\, d\mu(t), \tag{108}$$

$$K_2(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)) = \\ = \int\limits_{[0;\ 0.16\pi]} \sin\left(0.3x_1x_2x_3x_4t - \frac{\pi}{6}\right) d\mu(t), \tag{109}$$

$$K_3(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)) = \\ = \int\limits_{[0;\ 0.16\pi]} \sin\left(0.15x_1x_2x_3x_4t - \frac{\pi}{5}\right) d\mu(t), \tag{110}$$

$$K_4\left(x_1(t),\ x_2(t),\ x_3(t),\ x_4(t)\right) = \\ = \int\limits_{[0;\ 0.16\pi]} \sin\left(0.54 x_1 x_2 x_3 x_4 t - \frac{\pi}{4}\right) d\mu(t). \tag{111}$$

The players are forced to use pure strategies $\{x_i(t)\}_{i=1}^{4}$ such that

$$x_1(t) \in \{2 + 0.5 \cdot (m_1 - 1)\}_{m_1=1}^{3} \subset [2;\ 3] \tag{112}$$

and

$$x_2(t) \in \{4 + 0.25 \cdot (m_2 - 1)\}_{m_2=1}^{4} \subset [4;\ 4.75] \tag{113}$$

and

$$x_3(t) \in \{1 + 0.5 \cdot (m_3 - 1)\}_{m_3=1}^{2} \subset [1;\ 1.5] \tag{114}$$

and

$$x_4(t) \in \{3 + 0.2 \cdot (m_4 - 1)\}_{m_4=1}^{3} \subset [3;\ 3.4], \tag{115}$$

and they can change their values only at time points

$$\left\{\tau^{(l)}\right\}_{l=1}^{7} = \{0.02 l \pi\}_{l=1}^{7}. \tag{116}$$

Consequently, this game can be thought of as it is defined on parallelepiped lattice

$$\{2 + 0.5 \cdot (m_1 - 1)\}_{m_1=1}^{3} \times \{4 + 0.25 \cdot (m_2 - 1)\}_{m_2=1}^{4} \times \\ \times \{1 + 0.5 \cdot (m_3 - 1)\}_{m_3=1}^{2} \times \{3 + 0.2 \cdot (m_4 - 1)\}_{m_4=1}^{3} \subset \\ \subset [2;\ 3] \times [4;\ 4.75] \times [1;\ 1.5] \times [3;\ 3.4], \tag{117}$$

that is this game is a succession of 8 finite $3 \times 4 \times 2 \times 3$ (quadmatrix) games

$$\left\langle \left\{ \left\{x_1^{(m_1-1)}\right\}_{m_1=1}^{3},\ \left\{x_2^{(m_2-1)}\right\}_{m_2=1}^{4},\ \left\{x_3^{(m_3-1)}\right\}_{m_3=1}^{2},\ \left\{x_4^{(m_4-1)}\right\}_{m_4=1}^{3} \right\}, \\ \{\mathbf{H}_{1l},\ \mathbf{H}_{2l},\ \mathbf{H}_{3l},\ \mathbf{H}_{4l}\} \right\rangle = \\ = \left\langle \left\{ \{2 + 0.5 \cdot (m_1 - 1)\}_{m_1=1}^{3},\ \{4 + 0.25 \cdot (m_2 - 1)\}_{m_2=1}^{4},\ \{1 + 0.5 \cdot (m_3 - 1)\}_{m_3=1}^{2}, \\ \{3 + 0.2 \cdot (m_4 - 1)\}_{m_4=1}^{3} \right\},\ \{\mathbf{H}_{1l},\ \mathbf{H}_{2l},\ \mathbf{H}_{3l},\ \mathbf{H}_{4l}\} \right\rangle \tag{118}$$

with first player's payoff matrices

$$\left\{\mathbf{H}_{1l} = \left[h_{1l\omega_1\omega_2\omega_3\omega_4}\right]_{3\times4\times2\times3}\right\}_{l=1}^{8}$$

whose elements are

$$h_{1lm_1m_2m_3m_4} =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_1\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_1\bigl(2 + 0.5\cdot(m_1-1),\ 4 + 0.25\cdot(m_2-1),\ 1 +$$

$$+0.5\cdot(m_3-1),\ 3 + 0.2\cdot(m_4-1),\ t\bigr) d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\bigl(0.2\cdot(2 + 0.5\cdot(m_1-1))\,(4 + 0.25\cdot(m_2-1))\,(1 +$$

$$+0.5\cdot(m_3-1))\,(3 + 0.2\cdot(m_4-1))\,t\bigr) d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\left(0.0025t\cdot(3+m_1)\,(15+m_2)\,(1+m_3)\,(14+m_4)\right) d\mu\,(t)$$

$$\text{for } \ l = \overline{1,\ 7} \tag{119}$$

and

$$h_{1,8m_1m_2m_3m_4} =$$

$$= \int\limits_{[0.14\pi;\ 0.16\pi]} \sin\left(0.0025t\cdot(3+m_1)\,(15+m_2)\,(1+m_3)\,(14+m_4)\right) d\mu\,(t), \tag{120}$$

with second player's payoff matrices

$$\left\{\mathbf{H}_{2l} = \left[h_{2l\omega_1\omega_2\omega_3\omega_4}\right]_{3\times 4\times 2\times 3}\right\}_{l=1}^{8}$$

whose elements are

$$h_{2lm_1m_2m_3m_4} =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_2\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_2\bigl(2 + 0.5\cdot(m_1-1),\ 4 + 0.25\cdot(m_2-1),\ 1 +$$

$$+0.5\cdot(m_3-1),\ 3 + 0.2\cdot(m_4-1),\ t\bigr) d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\bigl(0.3\cdot(2 + 0.5\cdot(m_1-1))\,(4 + 0.25\cdot(m_2-1))\,(1 +$$

$$+0.5\cdot(m_3-1))\,(3 + 0.2\cdot(m_4-1))\,t - \tfrac{\pi}{6}\bigr) d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\left(0.00375t\cdot(3+m_1)\,(15+m_2)\,(1+m_3)\,(14+m_4) - \frac{\pi}{6}\right) d\mu\,(t)$$

$$\text{for } \ i = \overline{1,\ 7} \tag{121}$$

and

$$h_{2,8m_1m_2m_3m_4} =$$

$$= \int\limits_{[0.14\pi;\ 0.16\pi]} \sin\left(0.00375t \cdot (3+m_1)(15+m_2)(1+m_3)(14+m_4) - \frac{\pi}{6}\right) d\mu(t), \quad (122)$$

with third player's payoff matrices

$$\left\{\mathbf{H}_{3l} = [h_{3l\omega_1\omega_2\omega_3\omega_4}]_{3\times4\times2\times3}\right\}_{l=1}^{8}$$

whose elements are

$$h_{3lm_1m_2m_3m_4} =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_3\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_3\big(2+0.5\cdot(m_1-1),\ 4+0.25\cdot(m_2-1),\ 1+$$

$$+0.5\cdot(m_3-1),\ 3+0.2\cdot(m_4-1),\ t\big)d\mu(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\big(0.15\cdot(2+0.5\cdot(m_1-1))(4+0.25\cdot(m_2-1))(1+$$

$$+0.5\cdot(m_3-1))(3+0.2\cdot(m_4-1))\,t - \tfrac{\pi}{5}\big)d\mu(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\left(0.001875t \cdot (3+m_1)(15+m_2)(1+m_3)(14+m_4) - \frac{\pi}{5}\right) d\mu(t)$$

$$\text{for } i = \overline{1,\ 7} \quad (123)$$

and

$$h_{3,8m_1m_2m_3m_4} =$$

$$= \int\limits_{[0.14\pi;\ 0.16\pi]} \sin\left(0.001875t \cdot (3+m_1)(15+m_2)(1+m_3)(14+m_4) - \frac{\pi}{5}\right) d\mu(t), \quad (124)$$

and with fourth player's payoff matrices

$$\left\{\mathbf{H}_{4l} = [h_{4l\omega_1\omega_2\omega_3\omega_4}]_{3\times4\times2\times3}\right\}_{l=1}^{8}$$

whose elements are

$$h_{4lm_1m_2m_3m_4} =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_4\left(x_1^{(m_1-1)},\ x_2^{(m_2-1)},\ x_3^{(m_3-1)},\ x_4^{(m_4-1)},\ t\right) d\mu(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} f_4\big(2+0.5\cdot(m_1-1)\,,\ 4+0.25\cdot(m_2-1)\,,\ 1+$$

$$+0.5\cdot(m_3-1)\,,\ 3+0.2\cdot(m_4-1)\,,\ t\big)d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\big(0.54\cdot(2+0.5\cdot(m_1-1))\,(4+0.25\cdot(m_2-1))\,(1+$$

$$+0.5\cdot(m_3-1))\,(3+0.2\cdot(m_4-1))\,t-\tfrac{\pi}{4}\big)d\mu\,(t) =$$

$$= \int\limits_{[0.02\cdot(l-1)\pi;\ 0.02l\pi)} \sin\left(0.00675t\cdot(3+m_1)\,(15+m_2)\,(1+m_3)\,(14+m_4)-\frac{\pi}{4}\right)d\mu\,(t)$$

$$\text{for }\ i=\overline{1,\ 7} \tag{125}$$

and

$$h_{4,8m_1m_2m_3m_4} =$$

$$= \int\limits_{[0.14\pi;\ 0.16\pi]} \sin\left(0.00675t\cdot(3+m_1)\,(15+m_2)\,(1+m_3)\,(14+m_4)-\frac{\pi}{4}\right)d\mu\,(t). \tag{126}$$

Each of the $3\times4\times2\times3$ quadmatrix games (118) with (119)—(126) is solved in pure strategies. It takes no longer than 1.2 seconds to obtain all the 8 interval solutions with an Intel Core i7 processor. Besides, each of the games has a single pure-strategy equilibrium on intervals

$$\{[0.02\cdot(l-1)\,\pi;\ 0.02l\pi)\}_{l=1}^{7}\,,\quad [0.14\pi;\ 0.16\pi]\,.$$

Consequently, there is a single equilibrium stack $x_n^*(t)\in X_n$ for the n-th player, where $x_n^*(t)$ takes on values $\{\alpha_{nl}^*\}_{l=1}^{8}$ only. It is shown player-wise in Figure 1. The respective players' payoffs

$$\Big\{K_1\left(\alpha_{1l}^*,\ \alpha_{2l}^*,\ \alpha_{3l}^*,\ \alpha_{4l}^*\right),\ K_2\left(\alpha_{1l}^*,\ \alpha_{2l}^*,\ \alpha_{3l}^*,\ \alpha_{4l}^*\right),\ K_3\left(\alpha_{1l}^*,\ \alpha_{2l}^*,\ \alpha_{3l}^*,\ \alpha_{4l}^*\right),$$

$$K_4\left(\alpha_{1l}^*,\ \alpha_{2l}^*,\ \alpha_{3l}^*,\ \alpha_{4l}^*\right)\Big\}_{l=1}^{8} = \{h_{1l}^*,\ h_{2l}^*,\ h_{3l}^*,\ h_{4l}^*\}_{l=1}^{8} \tag{127}$$

are presented in Figure 2 along with the polylines of payoff cumulative sums

$$\left\{\sum_{q=1}^{l}h_{1q}^*,\ \sum_{k=1}^{l}h_{2q}^*,\ \sum_{k=1}^{l}h_{3q}^*,\ \sum_{k=1}^{l}h_{4q}^*\right\}_{l=1}^{8} = \left\{h_{1\sum}^{(l)*},\ h_{2\sum}^{(l)*},\ h_{3\sum}^{(l)*},\ h_{4\sum}^{(l)*}\right\}_{l=1}^{8}. \tag{128}$$

The final payoffs of the players

$$\left\{\sum_{q=1}^{8}h_{1q}^*,\ \sum_{k=1}^{8}h_{2q}^*,\ \sum_{k=1}^{8}h_{3q}^*,\ \sum_{k=1}^{8}h_{4q}^*\right\} = \left\{h_{1\sum}^{(8)*},\ h_{2\sum}^{(8)*},\ h_{3\sum}^{(8)*},\ h_{4\sum}^{(8)*}\right\} \tag{129}$$

are highlighted in Figure 2 with circles. Note that payoff cumulative sums $h_{2\sum}^{(l)*}$ and $h_{4\sum}^{(l)*}$ are not increasing polylines. Contrary to this, cumulative sums $h_{1\sum}^{(l)*}$ and $h_{3\sum}^{(l)*}$

are increasing polylines. Generally speaking, payoff cumulative sums

$$\left\{ \left\{ \sum_{q=1}^{l} h_{iq}^{*} \right\}_{i=1}^{N} \right\}_{l=1}^{M} = \left\{ \left\{ h_{i \sum}^{(l)*} \right\}_{i=1}^{N} \right\}_{l=1}^{M} \tag{130}$$

do not have to be non-decreasing polylines.

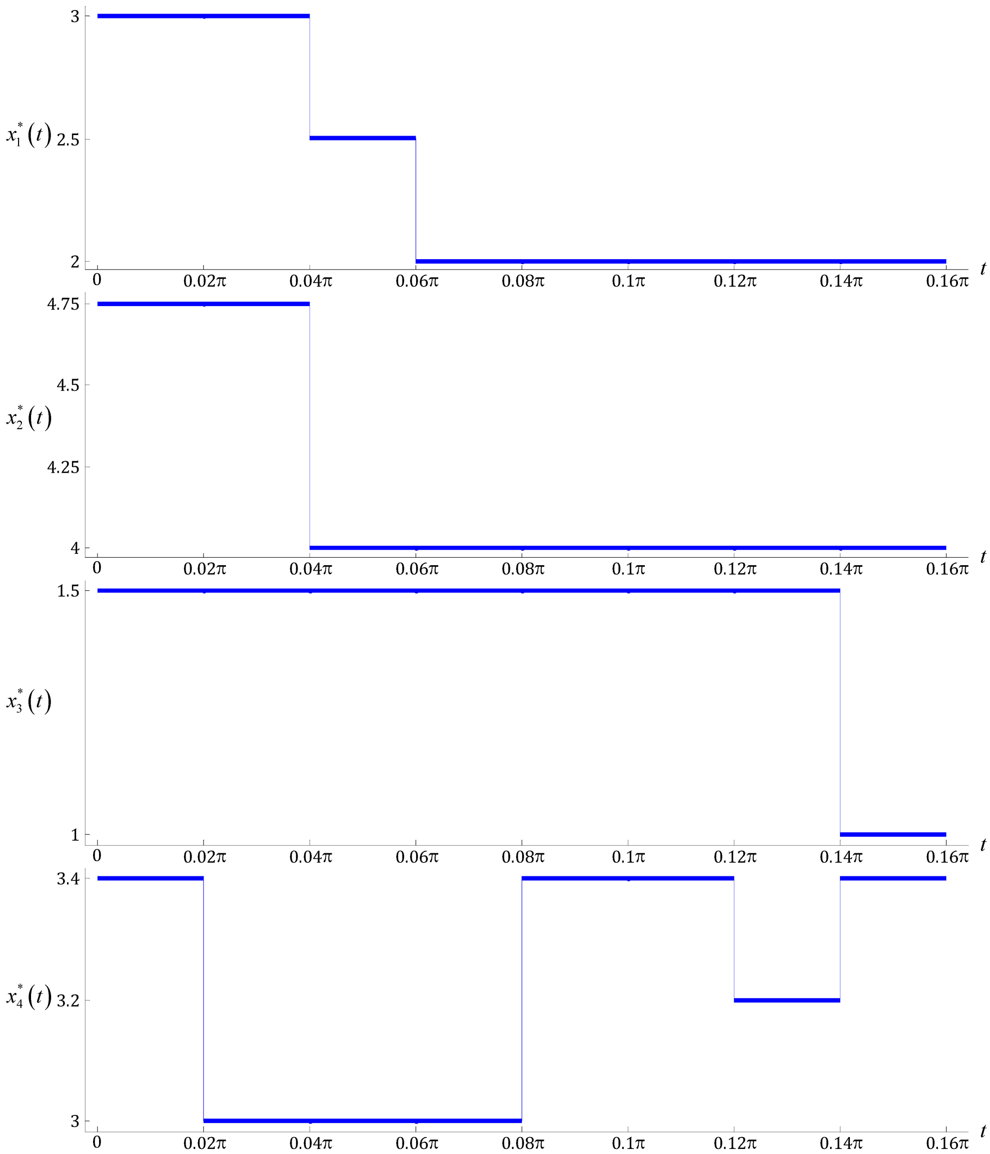

Figure 1: The players' pure-strategy equilibrium stacks in the game by (104) — (116)

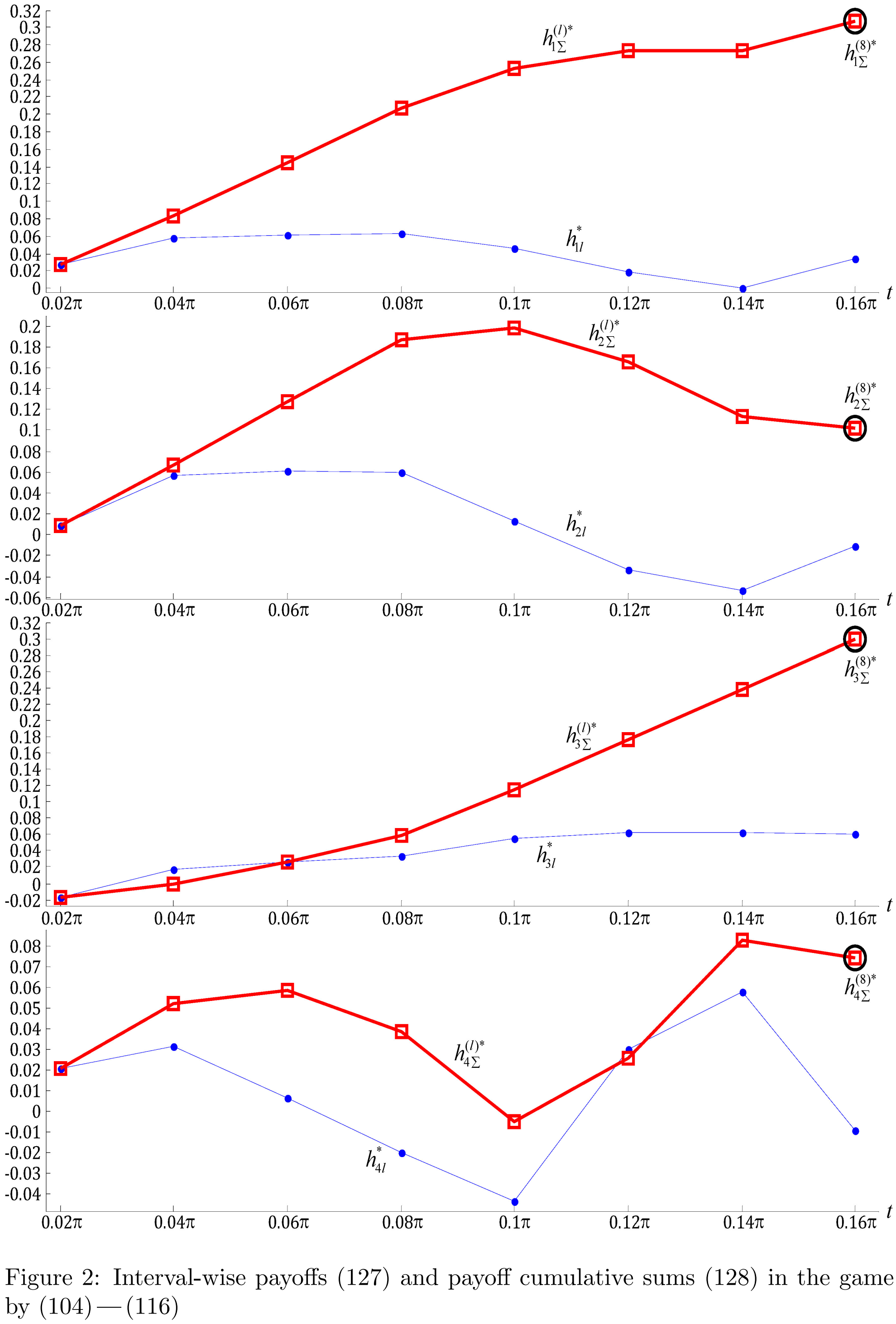

Figure 2: Interval-wise payoffs (127) and payoff cumulative sums (128) in the game by (104) — (116)

7. Discussion

The example clearly shows that solving a succession of multidimensional-matrix (quadmatrix in the considered example) games is far easier than tackling games whose players' pure strategies look like those staircase functions in Figure 1. Indeed, without solving the succession, the respective finite game by (104) — (116) defined on parallelepiped lattice (117) is rendered to a

$$6561 \times 65536 \times 256 \times 6561 \ \text{ staircase-function game.}$$

This quadmatrix game has 722204136308736 (more than 722 trillion) situations in pure strategies, which can hardly be handled in a reasonable computation time. By the way, the computation time has an exponential growth pattern as the size of the (hypercubic lattice) matrix increases.

Even if not every multidimensional-matrix game has a single equilibrium, a solution of the initial staircase-function game is built in the same way as (104) — (126). The only difference is that then there will be multiple stacked equilibria, which commonly induce instability of the players' behavior [23, 5, 14]. The time spent on computation of a stack depends on both the number of the player's pure strategies (on an interval) and the number of intervals. Stacking the "short" games' pure-strategy equilibria (by Theorem 5) is fulfilled trivially. When there is at least an equilibrium in mixed strategies for an interval (that actually falls within conditions of Theorem 6), the stacking is fulfilled as well implying that the resulting pure-mixed-strategy equilibrium in game (6) is realized successively, interval by interval, spending the same amount of time to implement both pure strategy and mixed strategy equilibria [18, 19].

The abovementioned behavior instability is a serious problem in non-cooperative games having multiple equilibria differing in the player's payoffs [22, 23, 15]. It is particularly solved by equilibria refinement with using domination efficiency along with maximin and the superoptimality rule [14]. The necessary condition is to have an asymmetry in the payoffs. The asymmetry allows distinguish more profitable (and thus stable) equilibria, whereupon the best equilibrium (equilibria) or equilibrium stacks are selected. Otherwise, they are not distinguishable.

Continuous games are ever struggled to be approximated or rendered to finite games so that their solutions could be easily implemented and practiced [10, 9, 11, 12, 15]. However, even a finite (that is, multidimensional-matrix) game may be not tractable due to gigantic number of situations in game. The presented method further "breaks" the initial staircase-function game with a purpose to obtain an equilibrium in a more reasonable time. So, the method is far more tractable than a straightforward approach to solving directly the staircase-function multidimensional-matrix game would be.

Here, the tractability does not depend on the number of (time) intervals. Unless the sets of possible values of players' pure strategies are of order of hundreds or thousands (when searching for equilibria in a "short" multidimensional-matrix game may take a few seconds and more), the method is entirely applicable. Moreover, the presented method is a significant contribution to the mathematical game theory and practice for avoiding too complicated solution approaches resulting from game continuities and functional spaces of pure strategies. This is similar to preventing Einstellung effect in modeling [16, 7]. The “breaking” of the staircase-function finite game into a succession of “short” multidimensional-matrix games herein “deeinstellungizes” such non-cooperative games.

A drawback is that a “short” multidimensional-matrix game may be intractable itself if its size is too big or there is a large number of players. The size limitation depends on requirements from the administrator, which, say, can limit the number of players to 3 or 4. If the interval breaking is over-thick, the “long” staircase-function multidimensional-matrix game may be solved in an unreasonable amount of time (although every “short” game is tractable and solved relatively fast). Consequently, the size of the “short” multidimensional-matrix game should be made as small as possible. The number of players should be necessarily limited.

8. Conclusion

A non-cooperative game defined on a product of staircase-function finite spaces is equivalent to a multidimensional-matrix game. In this game, a (pure) strategy is a complex set of simple actions ordinarily represented as a function of time. Players’ payoff matrices in this game are built very slowly, so it is impracticable to find any equilibria (as well as the other solution types) in such games using straightforwardly methods to solve a finite non-cooperative game. However, the multidimensional-matrix staircase-function game is equivalent to the succession of “short” multidimensional-matrix games, each defined on an interval where the pure strategy value is constant.

Owing to Theorem 6 along with Theorem 4 and Theorem 5 the equilibrium of the initial staircase-function game can be obtained by stacking the equilibria of the “short” multidimensional-matrix games. The stack is always possible, even when only time is discrete (and the set of pure strategy possible values is continuous). Any combination of the respective equilibria of the “short” multidimensional-matrix games is an equilibrium of the initial staircase-function game. Moreover, Theorem 5 allows finding a pure-strategy equilibrium of the initial (infinite or continuous) game by stacking the pure-strategy equilibria of the “short” (infinite or continuous) games.

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