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Pontryagin's Maximum Principle for Optimal Control Problems Governed by Nonlinear Impulsive Differential Equations

Hugo Leiva

ABSTRACT: In this paper, we derive the Pontryagin's maximum principle for optimal control problems governed by nonlinear impulsive differential equations. Our method is based on Dubovitskii-Milyutin theory, but in doing so, we assumed that the linear variational impulsive differential equation around the optimal solution is exactly controllable, which can be satisfied in many cases. Then, we consider an example as an application of the main result. After that, we study the case when the differential equation is of neutral type. Finally, several possible problems are proposed for future research where the differential equation, the constraints, the time scale, the impulses, etc. are changed.

In honor to Dr. Zoltan Varga

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1. Introduction

Pontriaguin's maximum (minimum) principle is used to optimize a functional depending on the state of the system and the best possible control that takes a dynamical system from one state to another, especially in the presence of constraints on state or input controls. It was formulated in 1956 by the Russian mathematician Lev Pontriaguin and his students(see [41]). It has as a special case the Euler-Lagrange equation of the calculus of variations. The result was first successfully applied to minimal time problems when input control is constrained, but it can also be useful in studying state constrained problems. In the following decades several abstract theories have been published to give a synthesis that would include different chapters of optimization, such as mathematical programming, classical variational calculus, and optimal control. The two most prominent theories are: Dubovitskii–Milyutin [16] and Iofee-Tihomirov [22]. In [22], the main result is a first order necessary condition for problems called "soft convex". The condition is formulated in terms of Lagrange's multipliers. In the Dubovitskii–Milyutin theory (which is applied in the present work) the fundamental idea is the following: Conic approximations are constructed to the data of an optimization problem with constrains, and in terms of duals elements of these cones, the optimality condition is expressed in the abstract Euler–Lagrange equation form. Given a class of optimization problems, the application of this theory consists in specifying the cones and their dual to express the Abstract Euler-Lagrange equation in terms of the problem in question. In the book of I. V. Girsanov [18] this method is carried out for several cases, such as the optimal control problem with a *finite number* of constraints on the state of the system. The main goal of this paper is to derive a general optimal condition (Pontryagin's maximum principle) for optimal control problems governed by impulsive differential equations. More specifically, we shall study the following problem

PROBLEM 1.1.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min \text{ loc.} \tag{1.1}$$

$$(x,\, u) \in E := \mathcal{PW}([0,T];\mathbb{R}^n) \times L^r_\infty([0,T];\mathbb{R}^r), \tag{1.2}$$

$$\dot{x}(t) = \varphi(x(t),\, u(t),\, t), \qquad x(0) = x_0 \tag{1.3}$$

$$x(T) = x_1;\; x_1,\, x_0 \in I\!R^n, \tag{1.4}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{1.5}$$

$$u(t) \in M, \quad t \in [0,\, T], \quad a.e., \tag{1.6}$$

where $0 < t_1 < t_2 < \cdots < t_p < T$, are fixed real numbers, $x \in \mathcal{PW}([0,T];\mathbb{R}^n)$, the control function u belongs to L^r_∞, $M \subset I\!R^r$ and the functions

$$\begin{aligned} \varphi &: I\!R^n \times I\!R^r \times [0,\, T] \longrightarrow I\!R, \\ \Phi &: I\!R^n \times I\!R^r \times [0,\, T] \longrightarrow I\!R^n, \\ \mathcal{J}_k &: I\!R^n \longrightarrow I\!R^n, \end{aligned}$$

where $\mathcal{PW}([0,T];\mathbb{R}^n)$ and L^r_∞ are define by

$$\begin{aligned}\mathcal{PW}([0,T];\mathbb{R}^n) &= \{z:[0,T]\to\mathbb{R}^n : z\in C(J';\mathbb{R}^n), \exists z(t_k^+), z(t_k^-)\\ &\quad \text{and} \quad z(t_k)=z(t_k^-), \quad k=1,2,\dots,p\},\end{aligned}$$

where $J=[0,T]$ and $J'=J\backslash\{t_1,t_2,\dots,t_p\}$, endowed with the norm

$$\|z\|_0 = \sup_{t\in[0,T]} \|z(t)\|_{\mathbb{R}^n},$$

and $L^r_\infty = L^r_\infty([0,T];\mathbb{R}^r)$ is the space of measurable function essentially bounded with the essential supremum norm.

For now these functions are smooth enough, so to prove the main results we will impose some additional conditions on the terms involved in the problem 1.1. The study of the controllability of differential equations with impulses is in effervescence at the moment, we can mention the following recent works on the controllability of such equations (see [8, 10, 28, 29, 30, 31, 32, 33, 36]), this in finite dimension, whereas in infinite dimension we can cite the following works ([2, 3, 5, 20, 38]). The Dubovitskii-Milyutin theory has been used to study optimal control problems for a long time, but not for impulsive differential equations, in this sense it is worthwhile to mention the work done in [9, 14, 15, 19, 21, 16, 24, 34]. Furthermore, we know there are a lot of works on optimal control problems using different techniques, for which one can see the research done in [23, 35, 37, 40] But, as far as we know, the optimal control problems for impulsive differential equations have not been studied much, only some particular works can be found in the literature, to mention some of them, we have the works carried out by ([1, 4, 6, 11, 26, 39, 42]).

Outline of the work: Section 2 contains preliminary results, here we summarizes the fundamental concepts and results of Dubovitskii-Milyutin theory that will be applied later; the intersection of cones lemma is presented. Then, the optimality condition in the abstract Euler–Lagrange equation form for a general optimization problem with constraints is formulated. To apply the general scheme of this theory to a specific class of problems, we must first compute the approximation cones. To do this, in subsections 2.3-2.5, we summarize and develop the methods to calculate the decay, admissible and tangent cones that appear in [18]. In addition, several extensions of these results are demonstrated, which facilitate the treatment for impulsive differential equations. In subsection 2.6, we present and prove some modifications of Minkowski-Farkas's Theorem, which simplify the explicit calculation of dual cones. Results of this subsection are useful to express the corresponding Euler-Lagrange equation to many problems in future investigations. In section 3, an optimal control problem governed by a nonlinear impulsive differential equation is considered. The main objective is to see that under certain conditions the impulses do not affect the optimality condition obtained by Pontryagin; roughly speaking, if the pulses are small enough, the maximum principle remains the same. In section 4, we prove that the necessary condition of optimality presented in Theorem 3.1 (maximum principle), under certain additional conditions, is also sufficient. To do this, we must assume

conditions that allow us to apply the general theorem of sufficient condition of optimality from the Dubovitskii-Milyutin theory, Theorem 2.17.
In section 5 we modify the optimal control problem by changing the boundary condition in its final state by placing a finite number of nonlinear constraints, and under certain conditions we again prove that the maximum principle persists.
In section 6 an example is presented as an application of these results obtained here.
In this section 7, we will show how Dubovitskii–Milyutin theory can be applied to generalize the Maximum principle of [18] to the case of optimal control problems governed by impulsive nonlinear neutral differential equations.
Finally, in section 8, we present several problems that could be solved in a similar way, which are part of future research.

2. Preliminaries Results

In this section, we summarize some fundamental results of the Dubovitskii- Milyutin theory. We formulate the general optimization problem with constraints and construct the approximation cones to the problem data (the objective function and restrictions), and the optimality condition in terms of the approximation cones dual is expressed by the Euler-Lagrange equation. The proof of these results can be refereed in [18].

2.1. Cones, Dual Cones and Dubovitskii–Milyutin Lemma

Let E be a locally convex topological linear space, and denote its dual space by E^*, the space of continuous linear functionals.

Definition 2.1. $K \subset E$ is a *cone with apex at zero* , if

$$\lambda K = K, \qquad (\lambda > 0).$$

Definition 2.2.

$$K^+ = \{f \in E^* \,/\, f(x) \geq 0, \quad \forall\, x \in K\},$$

is called the *dual cone* of K.

Proposition 2.3.

a) K^+ *is a* w^*- *closed and convex cone.*

b) $K^+ = (\overline{K})^+$, *(*$\overline{K}$ *is the* $w-$ *closure of* K*).*

c) $\left(\bigcup_{\alpha \in A} K_\alpha\right)^+ = \bigcap_{\alpha \in A} K_\alpha^+$ *where* A *is an index–set.*

d) If $K_1 \subset K_2$, *then* $K_2^+ \subset K_1^+$.

Definition 2.4. Let A be an arbitrary set and $K_\alpha \subset E, \;\; \alpha \in A$, be cones with apex at zero. Then, we define the following set

$$\sum_{\alpha \in A} K_\alpha^+ = \{f_{\alpha_1} + f_{\alpha_2} + \cdots + f_{\alpha_n}, \;\; f_{\alpha_i} \in K_{\alpha_i}^+, \;\; n \in \mathbb{N}, \;\; \alpha_i \in A \quad (i = 1, \ldots, n)\}.$$

Lemma 2.5. *Let $K_\alpha \subset E \quad (\alpha \in A)$ be convex cones $w-$closed, then*

$$\left(\bigcap_{\alpha \in A} K_\alpha \right)^+ = \overline{\sum_{\alpha \in A} K_\alpha^+} \qquad (w^* - \textit{closure} \;).$$

Lemma 2.6. *Let $K \subset E$ be a convex cone with apex at zero, $L \subset E$ a linear subspace such that $\overset{\circ}{K} \cap L \neq \emptyset$. Then $(K \cap L)^+ = K^+ + L^+$.*

Lemma 2.7. *Let $K_1, K_2, \ldots, K_n \subset E$ be open convex cones such that*

$$\bigcap_{i=1}^n K_i \neq \emptyset.$$

Then

$$\left(\bigcap_{i=1}^n K_i \right)^+ = \sum_{i=1}^n K_i^+.$$

Lemma 2.8. *(Dubovitskii–Milyutin). Let $K_1, K_2, \ldots, K_{n+1} \subset E$ be convex cones with apex at zero, with $K_1, K_2, \ldots, K_n$ open. Then*

$$\bigcap_{i=1}^{n+1} K_i = \emptyset$$

if and only if there are $f_i \in K_i^+ \quad (i = 1, 2, \ldots, n+1)$, not all zero such that

$$f_1 + f_2 + \cdots + f_n + f_{n+1} = 0.$$

2.2. The Abstract Euler–Lagrange Equation

Let us consider $F : E \longrightarrow I\!R$, and
$Q_i \subset E \quad (i = 1, 2, \ldots, n+1)$ such that the interior $\overset{\circ}{Q}_i \neq \emptyset \;\; (i = 1, 2, \ldots, n)$. Consider the following problem

$$F(x) \longrightarrow \min \textit{ loc} \tag{2.1}$$

$$x \in Q_i \qquad (i = 1, 2, \ldots, n+1). \tag{2.2}$$

Remark 2.9. The sets Q_i, $(i = 1, 2, \ldots, n)$ usually are given by constraints inequality type, and Q_{n+1} by constraints equality type, and in general the interior $\overset{\circ}{Q}_{n+1} = \emptyset$.

To study the above problem, we give some previous definitions and lemmas.

Definition 2.10. The vector $h \in E$ is a *vector of decay direction* of $F : E \longrightarrow I\!R$ at the point $x^\circ \in E$, if there exists a neighborhood U of the point x°, numbers $\alpha = \alpha(F, x^\circ, h) < 0$ and $\varepsilon_0 \in I\!R_+$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\overline{h} \in U$ the following inequality holds

$$F(x^\circ + \varepsilon \overline{h}) \leq F(x^\circ) + \varepsilon\, \alpha.$$

Lemma 2.11. *The decay vectors of F in x° generate an open cone with apex at zero which will be denoted by $K_d = K_d(F, x^\circ)$, and it will be called as decay cone.*

Next, we introduce similar definitions for different constraints of the problem. For a constraint of inequality–type, we give the following definition.

Definition 2.12. The vector $h \in E$ is an *admissible vector to $Q \subset E$ in the point* $x^\circ \in Q$, if there is a neighborhood U of the point x° and $\varepsilon_0 \in I\!R_+$, such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $\overline{h} \in U$, we have that

$$x^\circ + \varepsilon\, \overline{h} \in Q.$$

Lemma 2.13. *The admissible vectors to Q in x° generate an open cone with apex at zero, which will be denoted by $K_a := K_a(Q, x^\circ)$, and will be called admissible cone to Q in x°.*

To constraints of equality–type, we introduce the following definition.

Definition 2.14. The vector $h \in E$ is called a *tangent vector* to $Q \subset E$ at the point x°, if there are $\varepsilon_0 \in I\!R_+$ and a function $\theta : [0, \varepsilon] \longrightarrow E$, such that

$$\lim_{\varepsilon \to 0^+} \frac{\theta(\varepsilon)}{\varepsilon} = 0,$$

and

$$x^\circ + \varepsilon\, h + \theta(\varepsilon) \in Q \qquad (\varepsilon \in (0, \varepsilon_0)).$$

The set of all tangent vectors to Q in x° is a cone with apex at zero, which will be denoted by $K_T := K_T(Q, x^\circ)$; and will be called *tangent cone*.

Theorem 2.15. *(Dubovitskii–Milyutin). Let us consider the following problem*

$$\begin{cases} F(x) \longrightarrow \min\, loc \\ \\ x \in Q_i, \quad (i = 1, 2, \ldots, n+1). \end{cases} \tag{2.3}$$

Let $x^\circ \in E$ be a solution of problem (2.3), and suppose that:

a) K_0 is the decay cone of F in x°.

b) K_i are the admissible cones to Q_i in $x^\circ \in Q_i$ $(i = 1, 2, \ldots, n)$.

c) K_{n+1} is the tangent cone to Q_{n+1} in x°.

Then, if $K_i \quad (i = 0, 1, 2, \ldots, n + 1)$ are convex, there exist functions $f_i \in K_i^+$, $(i = 0, 1, \ldots, n + 1)$ not all zero such that

$$f_0 + f_1 + \cdots + f_{n+1} = 0 \tag{2.4}$$

Equation (2.4) is called the **Abstract Euler-Lagrange equation**.

Remark 2.16. Sometimes it is important to ensure that $f_0 \neq 0$; an examination of the proof of Theorem 2.15 shows that a sufficient condition for this is that

$$\bigcap_{i=1}^{n+1} K_i = \emptyset.$$

To apply the Dubovitskii–Milyutin theorem to specific problems, we must follow the following scheme:

i) Determine the decay vectors.

ii) Determine the admissible vectors.

iii) Determine the tangent vectors.

iv) Build the dual cones.

Next, we will face problems (i) - (iv). The necessary optimality condition stated in Theorem 2.15, under certain conditions, is also sufficient:

Theorem 2.17. *Suppose that the following conditions hold:*

α) F is continuous and convex,

β) Q_i is convex $(i = 1, 2, \ldots, n + 1)$,

γ) $\left(\bigcap_{i=1}^{n} \mathring{Q}_i\right) \cap Q_{n+1} \neq \emptyset$,

δ) $x^\circ \in \bigcap_{i=1}^{n+1} Q_i$,

ε) $K_i \quad (i = 0, 1, \ldots, n + 1)$ are defined as in Theorem 2.15.

Then, x° is a solution of the problem (2.3) if and only if there exist $f_i \in K_i^+ \quad (i = 0, 1, 2, \ldots, n + 1)$ not all zero such that

$$f_0 + f_1 + f_2 + \cdots + f_{n+1} = 0.$$

2.3. Cones of Decay Vectors

In this subsection we explicitly compute the cones of decay vectors for several functions.

Definition 2.18. Let E be a linear space and $F : E \longrightarrow I\!\!R$ a function. Then, we shall say that F has directional derivative in $x^\circ \in E$ on the direction of $h \in E$ if the following limit there exists:

$$\lim_{\varepsilon \to 0^+} \frac{F(x^\circ + \varepsilon\, h) - F(x^\circ)}{\varepsilon} =: F'(x^\circ,\, h). \tag{2.5}$$

For $x^\circ \in E$.

Theorem 2.19. *If $h \in K_d$ and there exists $F'(x^\circ,\, h)$, then $F'(x^\circ,\, h) < 0$.*

Theorem 2.20. *If E is a Banach space, F is locally Lipschitzian in x°, and $F'(x^\circ,\, h) < 0$, then $h \in K_d(F,\, x^\circ)$.*

Theorem 2.21. *(See [18, pg 45]). Let $F : E \longrightarrow I\!\!R$ be a continuous and convex function in a topological linear space E and $x^\circ \in E$, then F has directional derivative in all directions at x° and also we have that*

a) $F'(x^\circ,\, h) = \inf \left\{ \dfrac{F(x^\circ + \varepsilon\, h) - F(x^\circ)}{\varepsilon} / \varepsilon \in I\!\!R_+ \right\}$,

b) $K_d(F,\, x^\circ) = \{h \in E / F'(x^\circ,\, h) < 0\}$.

Theorem 2.22. *(See [18, pg 48]). If E is a Banach space and F is Fréchet–differentiable in $x^\circ \in E$, then*

$$K_d(F,\, x^\circ) = \{h \in E / F'(x^\circ)h < 0\}$$

where $F'(x^\circ)$ is the Fréchet's derivative of F in x°.

Example 2.23. In the same way as the example 7.3 of (See [18, pg 50]) we obtain the following result:

Let $E = \mathcal{PW}([0,T];\mathbb{R}^n) \times L^r_\infty[0,\, T]$ and $F : E \longrightarrow I\!\!R$ defined as follows

$$F(x,\, u) := \int_0^T \Phi(x(t),\, u(t),\, t)dt,$$

$\Phi : I\!\!R^n \times I\!\!R^r \times I\!\!R \longrightarrow I\!\!R$ is a continuous function in its first two variables and measurable in the third variable, and has a derivative in its first and second variables Φ_x and Φ_u respectively bounded. Then, we have that

$$\begin{aligned} F'(x^\circ,\, u^\circ)(x,\, u) &= \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt, \\ \text{and} \quad K_d(F,\, (x^\circ,\, u^\circ)) &= \{(x,\, u) \in E / F'(x^\circ,\, u^\circ)(x,\, u) < 0\}. \end{aligned}$$

2.4. Cones of Admissible Vectors

Let E be a topological linear space, $F : E \longrightarrow I\!R$ a continuous function, $x^{\circ} \in E$ and

$$Q = \{x \in E / F(x) \leq F(x^{\circ})\}.$$

Lemma 2.24. *Let $K_a = K_a(Q, x^{\circ})$ and $K_d = K_d(F, x^{\circ})$, then $K_d \subset K_a$.*

The proof of above Lemma is trivial. There are cases in which $K_d = K_a$.

Theorem 2.25. *(See [18, pg 58]) Suppose that*

i) There exists $F'(x^{\circ}, h) \quad (h \in E)$.

ii) There exists $\overline{h} \in E$ such that $F'(x^{\circ}, \overline{h}) < 0$.

iii) $F'(x^{\circ}, \cdot)$ is convex.

Then

$$K_a \subset \{h \in E \,/\, F'(x^{\circ}, h) < 0\} = K_d.$$

Theorem 2.26. *(See [18, pg 59]). If Q is an arbitrary convex set with $\overset{\circ}{Q} \neq \emptyset$, then*

$$K_a = \{h \in E / h = \lambda(x^{\circ} - x),\ x \in \overset{\circ}{Q},\ \lambda \in I\!R_{+}\}.$$

2.5. Cones of Tangent Vectors

In this section we basically mention the so-called Lusternik Theorem, which is a powerful tool for calculating the cone of tangent vectors.

Theorem 2.27. *(Lusternik). Let E_1, E_2 Banach spaces, and suppose that*

a) $x^{\circ} \in E_1,\ P : E_1 \longrightarrow E_2$ is Fréchet's differentiable in x° and $P(x^{\circ}) = 0$.

b) $P'(x^{\circ}) : E_1 \longrightarrow E_2$ is surjective.

Then the cone of tangent vectors K_T to the set $Q := \{x \in E_1 / P(x) = 0\}$ in the point $x^{\circ} \in Q$, is given by

$$K_T = Ker\ P'(x^{\circ}).$$

The proof of above theorem (which is not trivial) can be found in [22, pg 30].

2.6. Relationship Between Approximation Cones and Their Dual

In this subsection, we present results that establishes a closed relationship between approximation cones and their dual.

Theorem 2.28. *If K is a linear subspace of a topological linear space E, then*

$$K^{+} = \{f \in E^* \,/\, f(x) = 0, \quad \forall\, x \in K\} =: K^{\perp},$$

where $K^{\perp}$ is called the annihilator of K.

Theorem 2.29. *Let $f \in E^*$ and $K_1 := \{x \in E \,/\, f(x) = 0\}$,*
$K_2 := \{x \in E \,/\, f(x) \geq 0\}, K_3 := \{x \in E \,/\, f(x) > 0\}$. Then:

i) If $f \neq 0$, then $K_1^+ = \{\lambda f \,/\, \lambda \in I\!R\}, \quad K_2^+ = K_3^+ = \{\lambda f \,/\, \lambda \in I\!R_{+0}\}$.

ii) If $f = 0$, then $K_1^+ = \{0\}, \quad K_2^+ = \{0\}$ and $K_3^+ = E^*$.

The proof of Theorems 2.28 and 2.29 is trivial.

Theorem 2.30. *Let E be a topological linear space and $F : E \longrightarrow I\!R$ continuous and convex. For $x^\circ \in E$, let us consider the following set*

$$Q := \{x \in E \,/\, F(x) \leq F(x^\circ)\}.$$

Now, we define

$$Q^* := \{f \in E^* \,/\, f(x) \geq f(x^\circ), \quad (x \in Q)\}.$$

Then

i) $K_T^+(Q, x^\circ) = Q^*$,

ii) If there exists $\overline{x} \in E$ such that $F(\overline{x}) < F(x^\circ)$, then

$$K_d^+ = K_a^+ = K_T^+ = Q^*.$$

Proof. Let $f \in Q^*$ and $h \in K_T$; then, by definition of K_T, there are $\epsilon_0 \in I\!R_+$, and $\theta : [0, \varepsilon_0] \to E$ such that

$$\lim_{\varepsilon \to 0^+} \frac{\theta(\varepsilon)}{\varepsilon} = 0,$$

and

$$x^\circ + \varepsilon\, h + \theta(\varepsilon) \in Q, \quad (\varepsilon \in (0, \varepsilon_0)).$$

Therefore

$$f(x^\circ + \varepsilon\, h + \theta(\varepsilon)) \geq f(x^\circ), \quad (\varepsilon \in (0, \varepsilon_0)).$$

Then $f(h) \geq 0$. Hence $f \in K_T^+$, that is to say

$$Q^* \subset K_T^+.$$

Let $f \in K_T^+$ and $x \in Q$, then by the convexity of Q, we have that $x - x^\circ$ is a tangent vector to Q in the point x°, then it follows

$$f(x - x^\circ) \geq 0,$$

or equivalently $f \in Q^*$.

Therefore
$$K_T^+ = Q^*.$$

Suppose (ii) hods, i.e., there exists $\overline{x} \in E$ such that $F(\overline{x}) < F(x^\circ)$, this implies that there exists $\overline{h} \in E$ such that $F'(x^\circ, \overline{h}) < 0$. In fact

Let $\overline{h} = \overline{x} - x^\circ$. Then, since F is continuous and convex, it follows

$$\begin{aligned} F'(x^\circ, \overline{h}) &\leq F(x^\circ + \overline{h}) - F(x^\circ) \\ &= F(\overline{x}) - F(x^\circ) < 0. \end{aligned}$$

Now, let us see that $K_a \subset K_d$; by Theorem 2.21, we have that

$$K_d = \{h \in E \,/\, F'(x^\circ, h) < 0\}.$$

Let $h \in K_a$, then there is $\varepsilon_0 \in I\!R_+$ such that $x_0 + \varepsilon\, h \in Q$ for all $\varepsilon \in (0, \varepsilon_0)$, therefore $F(x^\circ + \varepsilon\, h) \leq F(x^\circ), \quad (\varepsilon \in (0, \varepsilon_0))$, which implies that $F'(x^\circ, h) \leq 0$. Since K_a is open, there is a neighbourhood U of h such that $U \subset K_a$. Then, for $\gamma \in I\!R_+$ small enough, we have that

$$h_\gamma := h + \gamma(h - \overline{h}) \in U.$$

Then

$$F'(x^\circ, h_\gamma) \leq 0 \quad \text{and} \quad h = \frac{1}{1+\gamma} h_\gamma + \frac{\gamma}{1+\gamma} \overline{h}.$$

Due to the fact that $F'(x^\circ, \cdot)$ is convex, we obtain that

$$F'(x^\circ, h) \leq \frac{1}{1+\gamma} F'(x^\circ, h_\gamma) + \frac{\gamma}{1+\gamma} F'(x^\circ, \overline{h}) < 0.$$

By Theorem 2.25, we have that $K_a \subset K_d$.

Let us prove that $K_a^+ = K_T^+$. In fact, condition (ii) implies that $\overset{\circ}{Q} \neq \emptyset$. Thus, by Theorem 2.26, it follows that

$$K_a = \{h \in E \,/\, h = \lambda(x - x^\circ), \quad x \in \overset{\circ}{Q}, \quad \lambda \in I\!R_+\}.$$

Let $f \in K_a^+$ and $x \in \overset{\circ}{Q}$, then $x - x^\circ \in K_a$, thus, we have that

$$f(x) \geq f(x^\circ) \qquad (x \in \overset{\circ}{Q}),$$

Given that F is continuous and convex, $\overline{\overset{\circ}{Q}} = \overline{Q} = Q$, we have that

$$f(x) \geq f(x^\circ) \qquad (x \in Q).$$

Therefore $f \in Q^*$, that is

$$K_a^+ \subset Q^* = K_T^+,$$

but $K_a \subset K_T$. Then

$$K_a^+ = K_T^+ = Q^*.$$

□

From the above proof, we have the following consequence

Corollary 2.31. *If F is convex and continuous, and there is $\overline{x} \in E$ such that $F(\overline{x}) < F(x^\circ)$, then*

$F'(x^\circ, h) < 0$ if and only if, there exists $\lambda \in \mathbb{R}_+,\ x \in E$ such that $F(x) < F(x^\circ)$ and $h = \lambda(x - x^\circ)$.

Theorem 2.32. *Let E_1, E_2 be topological linear spaces and $A : E_1 \longrightarrow E_2$ a linear operator. Let $E = E_1 \times E_2$ be the product space and consider*

$$K := G_A = \{x \in E \,/\, x = (x_1, x_2), \quad Ax_1 = x_2\}.$$

Then

$$K^+ = \{f \in E^*,\ f = (f_1, f_2) \,/\, f_1 = -A^* f_2\}.$$

The proof of above Theorem is trivial.

2.6.1. Minkowski-Farkas's Theorem and its Aplications

Theorem 2.33. *(Minkowski-Farkas see [18, pg 70]). Let E_1 and E_2 be topological linear spaces, and $K_2 \subset E_2$ a convex cone with apex at zero, and consider $A : E_1 \longrightarrow E_2$ a continuous linear operator. If we define*

$$K_1 := \{x_1 \in E_1 \,/\, Ax_1 \in K_2\},$$

and suppose that there exists $\overline{x}_1 \in E_1$ such that $A\overline{x}_1 \in \overset{\circ}{K}_2$, then

$$K_1^+ = A^* K_2^+.$$

Remark 2.34. Below we will give different versions of Minkowski-Farkas's Theorem. Before, we shall prove a known Lemma since part of its proof given here will be applied in the proof of Theorem 2.36.

Lemma 2.35. *Let E be a locally convex topological linear space, and A, B linear subspaces such that A is finite-dimensional, and B is closed. Then $A + B$ is also closed.*

Proof. Assume, without loss of generality, that $A \cap B = \{0\}$. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of A, then, since the space E is locally convex and $e_i \notin B$, $i = 1, 2, \ldots, n$, by Separation Theorem there are $g_i \in E^*$ $(i = 1, 2, \ldots, n)$ such that

$$\begin{aligned} g_i(e_i) &> 0 \qquad (i = 1, 2, \ldots, n) \\ g_i(x) &= 0 \qquad (i = 1, 2, \ldots, n;\ x \in B). \end{aligned}$$

We consider the following functionals

$$f_i = \frac{g_i}{g_i(e_i)} \qquad (i = 1, 2, \ldots, n).$$

Let's introduce the following operator

$$\begin{aligned} P &: \quad E \longrightarrow \mathbb{R}^n \cong A, \\ P &= \quad (f_1, f_2, \ldots, f_n). \end{aligned}$$

Then we have that

$$\begin{aligned} P(x) &= x \qquad (x \in A) \\ P(x) &= 0 \qquad (x \in B). \end{aligned}$$

Let $a_s + b_s \in A + B$ $(s \in S)$ be a generalized sequence such that $(a_s + b_s)$ converges to $z \in E$. The fact that P is continuous implies that $P(a_s + b_s)$ converges to $P(z)$, which implies that $(a_s) \to P(z)$, and given that A is closed $P(z) \in A$, and by the same reason $(b_s) \to z - P(z) \in B$, that is

$$z = P(z) + z - P(z), \quad P(z) \in A, \quad z - P(z) \in B.$$

□

Theorem 2.36. *Let E_1, E_2 topological linear spaces and $A : E_1 \longrightarrow E_2$ a continuous linear operator. Let $K_2 \subset E_2$ be a convex cone with apex at zero such that K_2^+ is finitely generated, and define*

$$K_1 := \{x_1 \in E_1 \,/\, Ax_1 \in K_2\}.$$

Then

$$(K \cap L)^+ = K^+ + L^+ \quad \textit{and} \quad K_1^+ = A^* K_2^+.$$

Proof. Let $E := E_1 \times E_2$, $K := E_1 \times K_2$ and $L = G_A$. By the hypotheses $K^+ = \{0\} \times K_2^+$ is closed and finite-dimensional, and since L^+ is a subspace, which is w^*- closed, then we claim that $K^+ + L^+$ is w^*-closed. In fact, since E_1^*, E_2^* are locally convex topological linear spaces with respect to the w^*-topology, it follows that

$E_1^* \times E_2^*$ is a linear locally convex topological linear space with the product-topology, then we can apply Lemma 2.35 by taking A the subspace generated by K^+, $B = L^+$ and $P : E^* \longrightarrow A$ in the same way as in lemma 2.35.

Let $a_s + b_s \in K^+ + L^+ = A + B$ $(s \in I)$ be a generalized convergent sequence to $z \in E^*$, then

$$a_s \to P(z) \quad \text{and} \quad b_s \to z - P(z).$$

Since $K^+,\ L^+$ are closed, we have that $P(z) \in K^+$ and $z - P(z) \in L^+$.

Now, by applying Lemma 2.5, we obtain that

$$(K \cap L)^+ = K^+ + L^+.$$

On the other hand, we have

$$\begin{aligned} K^+ &= \{(f_1,\, f_2) \in E^* \,/\, f_1 = 0,\ f_2 \in K_2^+\}, \\ L^+ &= \{(g_1,\, g_2) \in E^* \,/\, g_1 = -A^* g_2\}, \end{aligned}$$

by Theorem 2.32. Let $f_1 \in K_1^+$ and put $f := (f_1,\, 0)$. Then, $f \in (K \cap L)^+$. In fact, let $x \in (K \cap L)$. Hence $x = (x_1,\, Ax_1)$ and $Ax_1 \in K_2$, this implies that $x_1 \in K_1$ by definition of K_1, thus $f(x) = f_1(x_1) \geq 0$ for all $x \in (K \cap L)$, that is $f \in (K \cap L)^+$. Then, there exist

$$(0,\, h) \in K^+,\ h \in K_2^+,\ (g_1,\, g_2) \in L^+,\ g_1 = -A^* g_2,$$

such that

$$(f_1,\, 0) = (0,\, h) + (g_1,\, g_2),$$

which implies that $f_1 = g_1$ and $h + g_2 = 0$, and therefore $f_1 = A^* h,\ h \in K_2^+$. Thus

$$K_1^+ \subset A^* K_2^+.$$

This claim $A^* K_2^+ \subset K_1^+$ is trivial. $\square$

Theorem 2.37. *Let E_1, E_2 be topological linear spaces and $A : E_1 \longrightarrow E_2$ a continuous linear operator, and let $K_2 \subset E_2$ be a convex cone with apex at point zero. Let us define the following cone*

$$K_1 := \{x_1 \in E_1 \,/\, Ax_1 \in K_2\}.$$

Suppose that there are $g \in E_1^*$ and $h \in K_2^+$ such that

$$A^* h \neq 0,\ K_1 = \{x_1 \in E_1 \,/\, g(x_1) \geq 0\}.$$

Then

$$K_1^+ = A^* K_2^+.$$

Proof. The proof that $A^*K_2^+ \subset K_1^+$ is trivial. Let us see that $K_1^+ \subset A^*K_2^+$. In fact, since $A^*K_2^+ \subset K_1^+$, by Theorem 2.28 there is $\beta \in I\!R_+$ such that $A^*h = \beta\, g$. Now, let $f_1 \in K_1^+$, then there exists $\lambda_1 \in I\!R_+$, such that $f_1 = \lambda_1\, g$. Therefore

$$f_1 = A^*\left(\frac{\lambda_1}{\beta}h\right), \quad \frac{\lambda_1}{\beta}h \in K_2^+,$$

which implies that $K_1^+ \subset A^*K_2^+$. $\square$

Proposition 2.38. *Let E_1, E_2 be Banach spaces and $A : E_1 \longrightarrow E_2$ a continuous linear operator such that Im $A = E_2$, and a convex cone $K_2 \subset E_2$ with apex at zero. Now, we define K_1 as follows*

$$K_1 := \{x_1 \in E_1 \,/\, Ax_1 \in K_2\}.$$

Then

$$K_1^+ \subset Im\ A^*.$$

Proof. From the fact that $K_2^+ = \overline{K}_2^+$, we can assume without loss of generality that $0 \in K_2$; which implies that Ker $A \subset K_1$, then by item $d)$ from Proposition 2.3, we have that $K_1^+ \subset (\text{Ker}A)^+$. But $(\text{Ker } A)^+ = (\text{Ker } A)^\perp$, then from the factorization lemma from [22, pg 16], we get that $(\text{Ker } A)^\perp = \text{Im } A^*$. $\square$

Proposition 2.39. *Let E_1, E_2 be topological linear spaces and $A_i : E_1 \longrightarrow E_2$ $(i = 1, 2, \ldots, h)$ continuous linear operators, and consider $K_2 \subset E_2$ a convex cone with apex at zero. Let us define the following cones*

$$K_1 := \{x_1 \in E_1 \,/\, A_i x_1 \in K_2 \quad i = 1, 2, \ldots, n\}.$$

Suppose that there exists $\overline{x}_1 \in E_1$ such that $A_i\overline{x}_1 \in \overset{\circ}{K}_2$ $(i = 1, 2, \ldots, n)$. Then

$$K_1^+ = \left(\sum_{i=1}^n A_i\right)^* K_2^+.$$

Proof. Let us define the following cones

$$K_{1i} := \{x_1 \in E_1 \,/\, A_i x_1 \in K_2\}; \qquad i = 1, 2, \ldots, n.$$

Then by the continuity of A_i $(i = 1, 2, \ldots, n)$, we have that $\overline{x}_1 \in \overset{\circ}{K}_{1i}$, $(i = 1, 2, \ldots, n)$, which implies that $\left(\bigcap_{i=1}^n \overset{\circ}{K}_{1i}\right) =: K_3 \neq \emptyset$. So, by Lemma 2.7 it follows that

$$K_3^+ = \sum_{i=1}^n (\overset{\circ}{K}_{1i})^+.$$

We have that $K_3 \subset K_1$, which implies

$$K_1^+ \subset K_3^+ = \sum_{i=1}^{n} (\overset{\circ}{K}_{1i})^+ = \sum_{i=1}^{n} K_{1i}^+$$

Therefore $K_1^+ \subset \displaystyle\sum_{i=1}^{n} K_{1i}^+$ and $\displaystyle\sum_{i=1}^{n} K_{1i}^+ \subset K_1^+$, which implies that

$$K_1^+ = \sum_{i=1}^{n} K_{1i}^+.$$

But, from Theorem 2.33, we have that $K_{1i}^+ = A_i^* K_2^+$ $(i = 1, 2, \ldots, n)$, then

$$K_1^+ = \sum_{i=1}^{n} A_i^* K_2^+.$$

$\square$

To conclude this section, below we will see some applications of the Minkowski-Farkas's Theorem and its versions.

Proposition 2.40. *Let us consider* $E = \mathcal{PW}([0,T], I\!R^n)$ *and the following cone*

$$K = \{x \in E : x(T) = 0\}.$$

Then $f \in K^+$ *if, an only if, there is* $a \in I\!R^n$ *such that*

$$f(x) = \langle a, x(T) \rangle \qquad (x \in E).$$

Proof. The sufficiency is trivial. Let us prove the necessity. Define the operator $L : E \longrightarrow I\!R^n,\ L(x) := x(T),\ (x \in E)$, and consider $f \in K^+$. Then, Im $L = I\!R^n$ and Ker $L \subset$ Ker f, hence by The Factorization Lemma from (see [22, pg 15]), there is a linear-continuous function $g : I\!R^n \longrightarrow I\!R$ such that

$$f = g \circ L.$$

But it is well known that g has the following form

$$g(x) = \langle a, x \rangle, \qquad (x \in I\!R^n),$$

for some fixed $a \in I\!R^n$. Therefore

$$f(x) = \langle a, x(T) \rangle, \qquad (x \in E).$$

$\square$

Example 2.41. Let $A : [0, T] \longrightarrow I\!R^{n\times n}$ and $B : [0, T] \longrightarrow I\!R^{n\times r}$ be measurable and bounded functions; and consider the following linear control system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad t \in [0, T], a.e, && (2.6)\\ x(T) &= 0 && (2.7)\end{aligned}$$

where $(x,\, u) \in E_1 := \mathcal{PW}([0, T]; \mathbb{R}^n) \times L^r_\infty\,[0,\, T]$. Let us define the following cone

$$K_1 := \{(x,\, u) \in E_1 \,/\, (2.6) - (2.7) \quad \text{hold}\}\,.$$

Proposition 2.42. *If (2.6) is controllable, then dim $K_1^+ = n$, and also for all $f \in K_1^+$ there is $a \in I\!R^n$ such that*

$$f(x,\, u) = \left\langle a,\, \int_0^T [A(t)x(t) + B(t)u(t)]dt \right\rangle, \qquad ((x,\, u) \in E_1).$$

Proof. Let $E_2 := \mathcal{PW}([0, T]; \mathbb{R}^n)$ and $K_2 := \{x \in E_2 \,/\, x(T) = 0\}$, and define the following operator $\Lambda : E_1 \longrightarrow E_2$ as follows

$$\Lambda(x,\, u)(t) := \int_0^t [A(\tau)x(\tau) + B(\tau)u(\tau)]d\tau \qquad ((x,\, u) \in E_1,\ t \in [0,\, T]).$$

Then

$$K_1 = \{(x,\, u) \in E_1 \,/\, \Lambda(x,\, u) \in K_2\}.$$

Λ is a continuous linear operator and dim $K_2^+ = n$. In fact, the assertion for Λ is trivial. Let us see that dim $K_2^+ = n$; for which it is enough to see the following: $f_2 \in K_2^+$ if, an only if, there is $a \in I\!R^n$ such that

$$f_2(x) = \langle a,\, x(T)\rangle \qquad (x \in E_2).$$

This follows from Proposition 2.40.
Now. let $\{e_1,\, e_2, \ldots,\, e_n\}$ be the canonic basis of $I\!R^n$, and define the following linear functionals $\overline{f}_i : E_2 \to I\!R, \quad i = 1,\, 2, \ldots,\, n$

$$\overline{f}_i(x) = \langle e_i,\, x(T)\rangle\,, \qquad (x \in E_2).$$

Then, given $f_2 \in K_2^+$ there exists $a \in I\!R^n$ such that $f_2(x) = \langle a,\, x(T)\rangle$. On the other hand, we now that $a = \sum_{i=}^n a_i e_i$. Then,

$$f_2 = \sum_{i=1}^n a_i \overline{f}_i \qquad (a_i \in I\!R,\ i = 1,\, 2, \ldots,\, n).$$

Let us see that $\{\overline{f}_1,\, \overline{f}_2, \ldots,\, \overline{f}_n\}$ is a linearly independent family, for which we consider $\alpha_i \in I\!R$ $(i = 1,\, 2, \ldots,\, n)$ such that

$$\alpha_1 \overline{f}_1 + \alpha_2 \overline{f}_2 + \cdots + \alpha_n \overline{f}_n = 0.$$

Next, since (2.6) is controllable, then for each $e_i,\ i = 1, 2, \dots, n$, there is $(x_i, u_i) \in E_1 \quad (i = 1, \dots, n)$ such that

$$x(T) = \alpha_i e_i \quad (i = 1, 2, \dots, n).$$

Thus

$$\alpha_1 \overline{f}_1(x_i) + \cdots + \alpha_n \overline{f}_n(x_i) = \sum_{i=}^{n} \alpha_i^2 = 0,$$

which proves that $\{\overline{f}_1, \overline{f}_2, \dots, \overline{f}_n\}$ is linearly independent; therefore $\dim K_2^+ = n$. Then, by Theorem 2.36 (Minkowski-Farkas´s theorem version), we have that

$$K_1^+ = \Lambda^* K_2^+.$$

That is to say, for all $f_1 \in K_1^+$, there is $a \in I\!\!R^n$ such that

$$\begin{aligned} f_1(x, u) &= \langle a, \Lambda(x, u)(T) \rangle \\ &= \left\langle a \int_0^T [A(t)x(t) + B(t)u(t)]dt \right\rangle \qquad ((x, u) \in E_1). \end{aligned}$$

□

Proposition 2.43. *Let $A, \mathcal{J}_k : [0, T] \longrightarrow I\!\!R^{n \times n}, k = 1, 2, 3, \dots, p$ and $B : [0, T] \longrightarrow I\!\!R^{n \times r}$ be measurable and bounded matrix functions. Suppose the following impulsive linear system is controllable on $[0, T]$ for any $\bar{b} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_p) \in I\!\!R^{np} = (I\!\!R^n)^p$*

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), & t \in (0, \tau], \quad t \neq t_k \\ x(t_k) = \mathcal{J}_k(t_k)x(t_k^-) + \bar{b}_k, & k = 1, 2, 3, \dots, p., \end{cases} \tag{2.8}$$

where $(x, u) \in E_1 := \mathcal{PW}([0, T]; \mathbb{R}^n) \times L_\infty^r[0, T]$. Let us define the following cone

$$K_2 := \left\{ (x, u) \in E_1 \,/\, x(T) = 0, \quad x(t_k^+) - \mathcal{J}_k(t_k)x(t_k^-) = 0, \quad k = 1, 2, 3, \dots, p \right\}.$$

Then $\dim K_2^+ = n(p+1)$, and also for all $f \in K_2^+$ there is $a \in I\!\!R^{n(p+1)}$ such that

$$f(x, u) = \left\langle a, \quad (x(T), x(t_1) - \mathcal{J}_k(t_1)x(t_1^-), \dots, x(t_p) - \mathcal{J}_k(t_p)x(t_p^-)) \right\rangle, \quad (x, u) \in E_1.$$

Proof. Consider the following linear and continuous operator $\Lambda : E_1 \to I\!\!R^{n(1+p)}$ defined as follows

$$\Lambda(x, u) = \left(x(T), x(t_1) - \mathcal{J}_k(t_1)x(t_1^-), \dots, x(t_p) - \mathcal{J}_k(t_p)x(t_p^-) \right).$$

Since system (2.8) is controllable, then $\operatorname{Im} \Lambda = I\!\!R^{n(1+p)}$. Now, let $f \in K_2^+$, then $\ker \Lambda \subset \ker f$, and by the factorization lemma from (see [22, pg 15]), there is a linear-continuous function $g : I\!\!R^{n(1+p)} \longrightarrow I\!\!R$ such that

$$f = g \circ \Lambda.$$

But, it is well known that g has the following form

$$g(x) = \langle a,\, x\rangle, \qquad (x \in I\!R^{n(1+p)}),$$

for some fixed $a \in I\!R^{n(1+p)}$. Therefore

$$f(x,u) = \langle a, \quad (x(T), x(t_1) - \mathcal{J}_k(t_1)x(t_1^-), \dots, x(t_p) - \mathcal{J}_k(t_p)x(t_p^-))\rangle, \quad ((x,u) \in E_1).$$

Now. let $\{e_1, e_2, \dots, e_{n(p+1)}\}$ be the canonic basis of $I\!R^{n(p+1)}$, where $e_i = (e_{i,1}, e_{i,2}, \dots, e_{i,(p+1)})$, with $e_{i,k} \in I\!R^n$, and define the following linear functionals $\overline{f}_i : E_1 \to I\!R, \quad i = 1, 2, \dots, n(p+1)$,

$$\overline{f}_i(x) = \left\langle e_i,\, (x(T), x(t_1) - \mathcal{J}_1(t_1)x(t_1^-), \dots, x(t_p) - \mathcal{J}_p(t_p)x(t_p^-))\right\rangle, \; ((x,u) \in E_1).$$

Then, given $f_2 \in K_2^+$ there exists $a \in I\!R^{n(p+1)}$ such that

$$f_2(x,u) = \left\langle a,\, (x(T), x(t_1) - \mathcal{J}_1(t_1)x(t_1^-), \dots, x(t_p) - \mathcal{J}_p(t_p)x(t_p^-))\right\rangle.$$

On the other hand, we know that $a = \sum_{i=1}^{p+1} a_i e_i$. Then,

$$f_2 = \sum_{i=1}^{p+1} a_i \overline{f}_i \qquad (a_i \in I\!R,\; i = 1,\, 2, \dots,\, n(p+1)).$$

Let us see that $\{\overline{f}_1,\, \overline{f}_2, \dots,\, \overline{f}_{(p+1)}\}$ is a linearly independent family, for which we consider $\alpha_i \in I\!R$ $(i = 1,\, 2, \dots,\, p+1)$ such that

$$\alpha_1 \overline{f}_1 + \alpha_2 \overline{f}_2 + \cdots + \alpha_{n(p+1)} \overline{f}_{(p+1)} = 0.$$

Since, for any $\bar{b} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_p) \in I\!R^{np}$ the impulsive system (2.8) is controllable, then for each $e_i,\; i = 1,\, 2, \dots,\, n(p+1)$, with $e_i = (e_{i1}, e_{i2}, \cdots, e_{ip})$, there is $(x_i,\, u_i) \in E_1 \quad (i = 1, \dots,\, p+1)$ such that

$$x(T) = \alpha_i e_{i1}, \quad \text{and} \quad x(t_k) - \mathcal{J}_k(t_k)x(t_k^-) = \alpha_i e_{ik} \;\; (k = 2,\, 3, \dots,\, p+1).$$

Thus

$$\alpha_1 \overline{f}_1(x_1, u_1) + \cdots + \alpha_n \overline{f}_{(p+1}(x_{(p+1},\, u_{(p+1)}) = \sum_{i=}^{p+1} \alpha_i^2 = 0,$$

which proves that $\{\overline{f}_1,\, \overline{f}_2, \dots,\, \overline{f}_{(p+1)}\}$ is linearly independent; therefore dim $K_2^+ = n(p+1)$.

□

Now, we will give an important example related with support functionals.

Example 2.44. Let $M \subset I\!R^r$ and $Q := \{u \in L_\infty^r[0,\, T]\,/\,u(t) \in M,\ t \in [0,\, T], a.e.\}$ and consider $u^\circ \in Q,\ a \in L_1^r[0,\, T]$ and $f : L_\infty^r \longrightarrow I\!R$ defined as follows

$$f(u) := \int_0^T \langle a(t),\, u(t)\rangle dt, \qquad (u \in L_\infty^r).$$

Let us suppose that $f(u) \geq f(u^\circ) \quad (u \in Q)$, then for all $U \in M$ and almost all $t \in [0,\, T]$

$$\langle a(t),\, U - u^\circ(t)\rangle \geq 0.$$

For details of this example see [18, pg 76].

Finally, we have the well known formula for integrating by part in the Lebesgue Integral

Proposition 2.45. *(Integration by parts for Lebesgues integral)*
Let $f, g : [\alpha, \beta] \to I\!R$ be two differentiable function almost every well, such that $f'g, fg' \in L^1([\alpha, \beta], I\!R)$. The the following formula holds

$$\int_{[\alpha,\beta]} f'g d\mu = \lim_{t\to\beta} f(t)g(t) - \lim_{t\to\alpha} f(t)g(t) - \int_{[\alpha,\beta]} fg'd\mu.$$

3. Optimal Control Problem for Impulsive Differential Equations

In this section we will show how Dubovitskii–Milyutin theory can be applied to generalize the maximum principle of [18]. The generalization consists in admitting an finite number of impulses in the differential equation presented in the problem. We will also see that in a linear dynamics case, under certain additional conditions, the maximum principle is a sufficient condition for optimality. After that, we shall give an example that illustrates the applicability of the main result of this section.

3.1. Maximum Principle in the Space $\mathcal{PW}([0,T];\mathbb{R}^n) \times L_\infty^r$

Let $n,\, r \in \mathbb{N}$ and $T \in I\!R_+$, and consider the functions $\Phi,\, \varphi, \mathcal{J}_k$:

$$\begin{aligned} \varphi &: I\!R^n \times I\!R^r \times [0,\, T] \longrightarrow I\!R, \\ \Phi &: I\!R^n \times I\!R^r \times [0,\, T] \longrightarrow I\!R^n, \\ \mathcal{J}_k &: I\!R^n \longrightarrow I\!R^n, \end{aligned}$$

where $\mathcal{PW}([0,T];\mathbb{R}^n)$ and L_∞^r are define by

$$\begin{aligned} \mathcal{PW}([0,T];\mathbb{R}^n) &= \{z : [0,T] \to \mathbb{R}^n : z \in C(J';\mathbb{R}^n), \exists z(t_k^+), z(t_k^-) \\ &\quad \text{and} \quad z(t_k) = z(t_k^-), \quad k = 1,2,3\ldots,p\}, \end{aligned}$$

where $J = [0, T]$ and $J' = J \backslash \{t_1, t_2, \ldots, t_p\}$, endowed with the norm

$$\|z\|_0 = \sup_{t \in [0,T]} \|z(t)\|_{\mathbb{R}^n},$$

and $L^r_\infty = L^r_\infty([0, T]; \mathbb{R}^r)$ be the space of measurable function essentially bounded with essential norm.

Let us suppose that the following conditions are fulfilled

a) Φ, φ and $\mathcal{J}_k$ are continuous functions, with derivatives Φ_x, Φ_u, φ_x, φ_u, $\mathcal{J}'_k$ are bounded functions on compact sets of $I\!R^n \times I\!R^r \times [0, T]$.

b) $M \subset I\!R^r$ is convex and closed with $\overset{\circ}{M} \neq \emptyset$.

c) The following linear system is controllable

$$\dot{x}(t) = \varphi_x(x^\circ(t), u^\circ(t), t)x(t) + \varphi_u(x^\circ(t), u^\circ(t), t)u(t), \quad t \in (0, \tau], \ a.e. \tag{3.1}$$

d) The corresponding impulsive linear variational equations around the point $(x^\circ, u^\circ) \in E$ is controllable on $[0, T]$ for any $\bar{b} = (\bar{b}_1, \bar{b}_2, \ldots, \bar{b}_p) \in (I\!R^n)^p$

$$\begin{cases} \dot{x}(t) = \varphi_x(x^\circ(t), u^\circ(t), t)x(t) + \varphi_u(x^\circ(t), u^\circ(t), t)u(t), \quad t \in (0, T], \quad t \neq t_k \\ x(t_k^+) = \mathcal{J}'_k(x^0(t_k^-))x(t_k^-) + \bar{b}_k, \quad k = 1, 2, 3, \ldots, p., \end{cases} \tag{3.2}$$

Remark 3.1. *According to the results presented in the references [10, 28, 29, 30, 31, 32, 33]) on the controllability of control systems governed by impulsive differential equations, a sufficient condition for system (3.2) to be controllable is that system (3.1) is controllable and the following condition holds for the impulses.*

$$\|\mathcal{J}'_k(x^0(t_k^-))\| < \frac{1}{p}, \quad k = 1, 2, 3, \ldots, p. \tag{3.3}$$

Theorem 3.1. *Suppose that conditions a) - d) are fulfilled. Let $(x^\circ, u^\circ) \in E$ be a solution of Problem 1.1:*
Then, there exists $\lambda_0 \in I\!R_{+0}$ and a function $\psi \in \mathcal{PW}([0, T]; \mathbb{R}^n)$ such that λ_0 and ψ both are different from zero, and ψ is solution of the following differential equation

$$\begin{cases} \dot{\psi}(\tau) = -\varphi_x^*(x^\circ(\tau), u^\circ(\tau), \tau)\psi(\tau) + \lambda_0 \Phi_x(x^\circ(\tau), u^\circ(\tau), \tau), \\ \psi(T) = a. \end{cases} \tag{3.4}$$

Moreover, for all $U \in M$ and almost all $t \in [0, T]$ the following inequality hols

$$\langle -\varphi_u^*(x^\circ(t), u^\circ(t), t)\psi(t) + \lambda_0 \, \Phi_u(x^\circ(t), u^\circ(t), t), U - u^\circ(t) \rangle \geq 0 \tag{3.5}$$

Proof. Let $\overline{F} : E \longrightarrow I\!R$ be a function defined as follows

$$\overline{F}(x,\, u) = \int_0^T \Phi(x(t),\, u(t),\, t)dt,$$

and let $Q := Q_1 \cap Q_2$ where Q_2, Q_1 are given by points $(x,\, u) \in E$, which satisfy (1.3)-(1.5) and (1.6) respectively.
Then, Problem 1.1 is equivalent to

$$\begin{cases} \overline{F}(x,\, u) \longrightarrow \min \\ (x,\, u) \in Q. \end{cases}$$

a) Analysis of the function $\overline{F}$.

Let $K_0 := K_d(F,\, (x^\circ,\, u^\circ))$ be the decay cone of $\overline{F}$ in the point $(x^\circ,\, u^\circ)$. Then, by Theorem 2.22, we have that

$$K_0 = \{(x,\, u) \in E \,/\, \overline{F}(x^\circ,\, u^\circ)(x,\, u) < 0\}.$$

Suppose for a moment that $K_0 \neq \emptyset$, then by Theorem 2.29 we obtain

$$K_0^+ = \{-\lambda_0\, \overline{F}(x^\circ,\, u^\circ) \,/\, \lambda_0 \in I\!R_{+0}\}.$$

By example 2.23, we obtain that

$$\overline{F}'(x^\circ,\, u^\circ)(x,\, u) = \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt, \qquad ((x,\, u) \in E).$$

Therefore, for all $f_0 \in K_0^+$, there exists $\lambda_0 \in I\!R_{+0}$ such that

$$f_0(x,\, u) = -\lambda_0 \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt, \qquad ((x,\, u) \in E).$$

b) Analysis of constraint Q_1.

Let us consider the set

$$Q_1' := \{u \in L_\infty^r[0,\, T] \,/\, u(t) \in M, \quad t \in [0,\, T], \quad a.e.\}.$$

Then $Q_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times Q_1'$. Moreover, by the hypothesis M is convex and closed, with $\overset{\circ}{M} = \emptyset$. So, the following statements hold

i) Q_1, Q_1' are convex and closed.

ii) $\overset{\circ}{Q}_1 \neq \emptyset$, $\quad \overset{\circ}{Q'}_1 \neq \emptyset$.

If we call K_1 the admissible cone to Q_1 in $(x^\circ,\, u^\circ) \in Q_1$, then

$$K_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times K_1',$$

where K_1' is the admissible cone Q_1' in $u^\circ \in Q_1'$.

Therefore, for all $f_1 \in K_1^+$ there is $f_1' \in K_1'^+$ such that $f_1 = (0,\, f_1')$.

By Theorem 2.26, it follows that f_1' is a support of Q_1' at u°.

c) Analysis of the constraint Q_2.

Let us find the tangent cone to Q_2 at the point $(x^\circ,\, u^\circ)$

$$K_2 := K_T(Q_2,\, (x^\circ,\, u^\circ)).$$

Consider the space $E_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times I\!R^{n(1+p)} = E_2$ and the operator: $P : E_1 \to E_2$ defined by

$$P(x,u)(t) = \left(x(t) - x_0 - \int_0^t \varphi(x(l),u(l),l)dl, \quad S(x,u), \quad x(T) - x_1 \right),$$

where

$$S(x,u) = \left(x(t_1) - \mathcal{J}_1(x(t_1^-)), x(t_2) - \mathcal{J}_2(x(t_2^-)) \cdots, x(t_p) - \mathcal{J}_p(x(t_p^-)) \right).$$

Then

$$P'(x^0,u^0)(\overline{x},\overline{u}) =$$
$$\left(\overline{x}(t) - \int_0^t (\varphi_x(x^\circ(l),\, u^\circ(l),\, l)\overline{x}(l) + \varphi_u(x^\circ(l),\, u^\circ(l),\, l)\overline{u}(l))dl, \quad S'(\overline{x},\overline{u}), \quad \overline{x}(T) \right),$$

with

$$S'(\overline{x},\overline{u}) = \left(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-) \right).$$

We want to find conditions under which the operator $P'(x^0,u^0)$ is onto in order to apply Lustenik theorem 2.27. So, for $(a(\cdot), b_1, b_2, \ldots, b_p, x_1) \in E_2$, we want to solve the equation

$$P'(x^0,u^0)(\overline{x},\overline{u}) = (a(\cdot), b_1, b_2, \ldots, b_p, x_1).$$

Now, suppose that $\overline{u} = 0$. Then, from ([25], pg 89), we know that the following Volterra integral equation

$$z(t) = a(t) + \int_0^t (\varphi_x(x^\circ(l),\, u^\circ(l),\, l)z(l)dl,$$

has a solution $z \in \mathcal{PW}([0,T];\mathbb{R}^n)$.
Next, since the impulsive linear variational equation (3.2) is controllable, for a point $(\bar{b}_1, \bar{b}_2, \dots, \bar{b}_p) \in I\!R^{np}$ with

$$\bar{b}_k = b_k - z(t_k) + \mathcal{J}_k'(x^0(t_k^-))z(t_k^-), \quad k = 1,2,3,\dots,p,$$

there exists a control $\overline{u} \in L_\infty^r$ such that the corresponding solution $y(t)$ of (3.2) satisfies $y(T) = x_1 - z(T)$ and

$$y(t_k) = \mathcal{J}_k'(x^0(t_k^-))y(t_k^-) + \bar{b}_k, \quad k = 1,2,3,\dots,p.$$

Let us make the following change of variable $\overline{x} = y + z$. then

$$\begin{aligned}
&P'(x^0,u^0)(\overline{x},\overline{u})(t) = ((y+z)(t) - \\
&\int_0^t (\varphi_x(x^\circ,u^\circ,l)(y+z)(l) + \varphi_u(x^\circ,u^\circ,l)\overline{u}(l))dl, \quad S'(\overline{x},\overline{u}), \quad (y+z)(T)\Big) \\
&= (y(t) + a(t) - \\
&\int_0^t (\varphi_x(x^\circ,\, u^\circ,\, l)y(l) + \varphi_u(x^\circ,\, u^\circ,l)\overline{u}(l))dl, \quad S'(\overline{x},\overline{u}), \quad (y+z)(T)\Big) \\
&= (a(t), \quad S'(\overline{x},\overline{u}), \quad x_1)\,.
\end{aligned}$$

Now, we shall see that $S'(\overline{x},\overline{u}) = (b_1, b_2, \dots, b_p)$. In fact,

$$\begin{aligned}
&S'(\overline{x},\overline{u}) = \\
&\big(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-)\big) = \\
&\big((y+z)(t_1) - \mathcal{J}_1'(x^0(t_1^-))(y+z)(t_1^-), \cdots, (y+z)(t_p) - \mathcal{J}_p'(x^0(t_p^-))(y+z)(t_p^-)\big) \\
&= \big(\bar{b}_1 + z(t_1) - \mathcal{J}_1'(x^0(t_1^-))z(t_1^-), \cdots\cdots, \bar{b}_p + z(t_p) - \mathcal{J}_p'(x^0(t_p^-))z(t_p^-)\big) \\
&= (b_1, b_2, \dots, b_p)\,.
\end{aligned}$$

Therefore, the operator $P'(x^0,u^0)$ is onto. Then, applying Lusternik's Theorem 2.27, we get that tangent cone K_2 is given by

$$K_2 = \{(x,\, u) \in E_1 \,/\, P'(x^\circ,\, u^\circ)(\overline{x},\overline{u}) = 0\}.$$

i.e., K_2 is the set of points $(\overline{x},\overline{u}) \in E_1$ such that

$$\begin{aligned}
\dot{\overline{x}}(t) &= \varphi_x(x^\circ(t),\, u^\circ(t),\, t)\overline{x}(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \quad t \neq t_k && (3.6)\\
\overline{x}(t_k^+) &= \mathcal{J}_k'(x^0(t_k^-))\overline{x}(t_k^-), \quad k = 1,2,3,\dots,p. && (3.7)\\
\overline{x}(T) &= 0 && (3.8)
\end{aligned}$$

Consider the following linear subspaces

$$L_1 = \{(\overline{x},\overline{u}) \in E_1/(3.6) - (3.7) \quad \text{hold}\}, \quad L_2 = \{(\overline{x},\overline{u}) \in E_1/ \quad \overline{x}(T) = 0\}.$$

Then, $K_2 = L_1 \cap L_2$. Now, let us compute K_2^+. By Proposition 2.40, we have that $f_{22} \in L_2^+$ if, and only if, there exists $a \in I\!R^n$ such that

$$f_{22}(x,\, u) = \langle a,\, x(T)\rangle \qquad ((x,\, u) \in E).$$

Moreover, the controllability of systems (3.1) - (3.2) implies that $L_1 + L_2$ is closed, then it follows that $L_1^+ + L_2^+$ is w^*- closed; hence by Lemma 2.5 we obtain that

$$K_2^+ = L_1^+ + L_2^+.$$

Since L_1 is a linear subspace, it follows from Theorem 2.28 that, for any

$f_{21} \in L_1^+$, $f_{21}(\overline{x}, \overline{u}) = 0$ for all $(\overline{x}, \overline{u})$ satisfying (3.6)-(3.7).

e) Euler-Lagrange equation.

It is easy to see that K_0, K_1, K_2, are convex cones. Hence, by Theorem 2.15 there are functionals $f_i \in K_i^+ \quad (i = 0,\, 1,\, 2,)$ not all zero such that

$$f_0 + f_1 + f_2 = f_0 + f_1 + f_{21} + f_{22} = 0. \tag{3.9}$$

Equation (3.9) can be written in the following form

$$-\lambda_0 \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt +$$
$$f_1'(x,\, u) + f_{21}(x,\, u) + \langle a,\, x(T)\rangle = 0, \quad ((x,\, u) \in E).$$

Now, for all $u \in L_\infty^r$ there exists $\overline{x}$, solution of system (3.6)-(3.7) with $\overline{x}(0) = 0$. Then $(\overline{x},\, u) \in L_1$. Therefore $f_{21}(\overline{x},\, u) = 0$.
Let ψ be the solution of equation (3.4), that is

$$\begin{cases} \dot{\psi}(t) = -\varphi_x^*(x^\circ(\tau),\, u^\circ(\tau),\, \tau)\psi(\tau) + \lambda_0 \Phi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau) \\ \psi(T) = a. \end{cases}$$

Multiplying both sides of this equation by $\overline{x}$ and integrating from 0 to T, we get

$$\lambda_0 \int_0^T \Phi_x(x^\circ,\, u^\circ,\, t)\overline{x}(t)dt - \langle a,\, x(T)\rangle =$$
$$\int_0^T \langle \dot{\psi}(t),\, \overline{x}(t)\rangle dt + \int_0^T \langle \varphi_x^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{x}(t)\rangle dt$$
$$-\langle a,\, \overline{x}(T)\rangle = \langle \psi(t),\, \overline{x}(t)\rangle]_0^T - \int_0^T \langle \psi(t),\, \dot{\overline{x}}(t)\rangle dt$$
$$+ \int_0^T \langle \varphi_x^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{x}(t)\rangle dt - \langle a,\, \overline{x}(T)\rangle = \langle \psi(T),\, \overline{x}(T)\rangle - \langle \psi(0),\, \overline{x}(0)\rangle$$
$$-\langle a,\, \overline{x}(T)\rangle + \int_0^T \langle \psi(t), \varphi_x(x^\circ,\, u^\circ,\, t)\overline{x}(t) - \dot{\overline{x}}(t)\rangle dt =$$
$$-\int_0^T \langle \psi(t), \varphi_u(x^\circ,\, u^\circ,\, t)\overline{u}(t)\rangle dt = -\int_0^T \langle \varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{u}(t)\rangle dt.$$

Then, from Euler-Lagrange equation (3.9), we obtain for ($u \in L^r_\infty[0, T]$), that

$$f_1'(t) = \int_0^T \langle -\varphi_u^*(x^\circ(t), u^\circ(t), t)\psi(t) + \lambda_0 \Phi_u(x^\circ(t), u^\circ(t), t)u(t)\rangle dt. \quad (3.10)$$

Since f_1' is a support of Q_1' at the point $u^\circ \in Q_1'$, from example 2.44, it follows that

$$\langle -\varphi_u^*(x^\circ(t), u^\circ(t), t)\psi(t) + \lambda_0 \Phi_u(x^\circ(t), u^\circ(t), t), U - u^\circ(t)\rangle \geq 0,$$

for all $U \in M$ and almost all $t \in [0, T]$.
Now, we will see that the case $\lambda_0 = 0,\ \psi = 0$, is not possible. In fact

If $\psi = 0$, then $\psi(T) = a = 0$. Thus

$$f_{22}(x, u) = \langle a, x(T)\rangle = 0 \qquad ((x, u) \in E),$$

that is $f_{22} \equiv 0$. So, from the fact that $\lambda_0 = 0$, we get that $f_0 = 0$. Also, from (3.10), we have that $f_1'(u) = 0 \quad (u \in L^r_\infty[0, T])$; then from Euler– Lagrange equation it follows that $f_{21} = 0$, where

$$f_2 = f_{21} + f_{22} = 0,$$

which contradicts Theorem 2.15.
So far, we have two additional assumptions:
Firstly, we assumed that $K_0 \neq \emptyset$, and secondly, we assumed that system

$$\dot{x} = \varphi_x(x^\circ, u^\circ, t)x(t) + \varphi_u(x^\circ, u^\circ, \ , t)u(t)$$

is controllable.
Now, we will prove, that these assumptions are superfluous. In fact, if $K_0 = \emptyset$, then by definition of K_0, we have that

$$\int_0^T [\Phi_x(x^\circ(t), u^\circ(t), t)x(t) + \Phi_u(x^\circ(t), u^\circ(t), t)u(t)]dt = 0 \quad ((x, u) \in E).$$

Let us put $\lambda_0 = 1,\ \psi(T) = a = 0$, then, from last computation, we have that

$$\int_0^T \langle \Phi_x^*(x^\circ, u^\circ, t)\psi(t), x(t)\rangle dt = -\int_0^T \langle \varphi_u^*(x^\circ, u^\circ, t)\psi(t), u(t)\rangle dt,$$

for all (x, u) such that x is solution of equation the (3.6)-(3.7). Then

$$\int_0^T \langle \varphi_u^*(x^\circ(t), u^\circ(t), t)\psi(t) + \Phi_u(x^\circ(t), u^\circ(t), t)u(t)\rangle dt = 0 \quad (u \in L^r_\infty[0, T])$$

which implies that

$$\langle -\varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t) + \Phi_u(x^\circ,\, u^\circ,\, t),\, U - u^\circ(t)\rangle = 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Now, suppose that system (3.1) is not controllable, then there is a non-trivial function $\psi \in C([0,\, T], I\!R^n)$ that is solution of

$$\dot{\psi}(t) = \varphi_x^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t),$$

such that, for all $t \in [0,\, T]$ it follows that

$$\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) = 0.$$

By taking $\lambda_0 = 0$, we get that ψ is solution of (3.4), and therefore

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t),\, U - u^\circ(t)\rangle \geq 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Thus, the proof of Theorem 3.1 is completed.

□

4. Sufficient Condition of Optimality

The necessary condition of optimality proved in Theorem 3.1 (Maximum Principle), under certain additional conditions, is also sufficient. In fact, let us consider the particular case of Problem 1.1 in which the differential equation is linear.

PROBLEM 4.1.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min$$

$$(x,\, u) \in E = \mathcal{PW}([0,T];\mathbb{R}^n) \times L_\infty^r([0,T];\mathbb{R}^r), \tag{4.1}$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{4.2}$$

$$x(0) = x_0, \quad x(T) = x_1; \qquad x_1,\, x_0 \in I\!R^n, \tag{4.3}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{4.4}$$

$$u(t) \in M, \qquad t \in [0,\, T],\, a.e. \tag{4.5}$$

where $A(\cdot) : [0,\, T] \longrightarrow I\!R^{n\times n}$, $B(\cdot) : [0,\, T] \longrightarrow I\!R^{n\times r}$ are measurable and bounded matrix functions and $\mathcal{J}_k - n \times n$ matrix, $k = 1, 2, 3, \ldots, p$. Let $(x^\circ,\, u^\circ) \in E$ be a point satisfying conditions (4.2)– (4.5).

Theorem 4.1. *Let us suppose that the conditions a) - d) from Theorem 3.1 are satisfied.*

Besides, let us assume the following hypotheses:

I) The system (4.2) and the impulsive system (4.2)-(4.4) are controllable.

II) There exists $\widetilde{u} \in L^r_\infty[0, T]$ *such that* $\widetilde{u}(t) \in \overset{\circ}{M}$, *for almost all* $t \in [0, T]$.

III) Φ *is a convex function in its two first variables.*

Then (x°, u°) *is global solution of Problem 4.1.*

Proof. Let us define the function $\overline{F} : E \longrightarrow I\!\!R$ as follows

$$\overline{F}(x, u) = \int_0^T \Phi(x(t), u(t), t)dt,$$

and the set $Q := Q_1 \cap Q_2$, where Q_2 is given by (4.2)-(4.4) and Q_1 by (4.5).
Then, Problem 4.1 is equivalent to:

$$\begin{cases} \overline{F}(x, u) \longrightarrow \min \\ (x, u) \in Q. \end{cases}$$

It is clear that $Q_i \quad (i = 1, 2)$ are convex sets, and from the condition $III)$ we have that $\overline{F}$ is convex, and from condition II) we have that $(\widetilde{x}, \widetilde{u}) \in \overset{\circ}{Q}_1 \cap Q_2$.
Thus, by Theorem 2.17 it follows:

(x°, u°) is a minimum point of F in Q if, and only if, there are $f_i \in K_i^+ \quad (i = 0, 1, 2)$, not all zero such that

$$f_0 + f_1 + f_2 = 0.$$

Here, $K_i \quad (i = 0, 1, 2)$ are cones defined as in Theorem 3.1. Now, suppose that the Maximum Principle of Theorem 3.1 holds. That is to say, there exist $\lambda_0 \in I\!\!R_{+0}$ and a function $\psi \in \mathcal{PW}([0, T]; \mathbb{R}^n)$ such that λ_0 and ψ are not both zero, and ψ is a solution of the following differential equation

$$\begin{cases} \dot{\psi}(t) = -A^*(t)\psi(t) + \lambda_0 \Phi_x(x^\circ(t), u^\circ(t), t) \\ \psi(T) = a \end{cases} \tag{4.6}$$

Moreover, for all $U \in M$ and almost all $t \in [0, T]$, we have that

$$\langle -B^*(t)\psi(t) + \lambda_0 \, \Phi_u(x^\circ(t), u^\circ(t), t), U - u^\circ(t)\rangle \geq 0. \tag{4.7}$$

Then, to prove the theorem, it is enough to see that there are $f_i \in K_i^+ \quad (i = 0, 1, 2)$ not all zero, such that $f_0 + f_1 + f_2 = 0$. To do so, we define the following functionals:

$$\begin{aligned} f_1' &: L_\infty^r \longrightarrow I\!R,\; f_1 : E \longrightarrow I\!R \\ f_1'(u) &:= \int_0^T \langle -B^*(t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, u(t)\rangle dt, \\ f_1 &= (0,\, f_1'). \end{aligned}$$

Let

$$Q_1' = \{u \in L_\infty^r \,/\, u(t) \in M, \quad t \in [0,\, T], \quad a.e.\}.$$

Then, from (4.7), we obtain

$$f_1'(u) \geq f_1'(u^\circ) \qquad (u \in Q_1').$$

Therefore f_1' is a support of Q_1' at u°. Hence $f_1 = (0,\, f_1') \in K_1^+$. Let us define the functional $f_{21} : E \longrightarrow I\!R$ as follows

$$f_{21}(x,\, u) := \lambda_0 \int_0^T [\Phi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)]dt$$
$$-f_1'(u) - \langle a,\, x(T)\rangle.$$

Now, we will see that $f_{21} \in L_1^+$, where

$$L_1 = \{(x,\, u) \,/\, (4.2), (4.4) \quad \text{hold}\},$$

as in the Theorem 3.1. In fact, suppose that $(x,\, u) \in L_1$, then multiplying both sides of the equation (4.6) by $\dot{x}$ and integrating by parts from 0 to T, we obtain that

$$\lambda_0 \int_0^T \langle \Phi_x(x^\circ(t),\, u^\circ(t),\, t)\psi(t), x(t)\rangle dt$$

$$-\langle a,\, x(T)\rangle = -\int_0^T \langle B^*(t)\psi(t),\, u(t)\rangle dt.$$

Then

$$f_{21}(x,\, u) = -f_1'(u) - \int_0^T \langle B^*(t)\psi(t),\, u(t)\rangle dt + \lambda_0 \int_0^T \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)dt.$$

Therefore

$$f_{21}(x,\, u) = -f_1'(u) + f_1'(u) = 0,$$

Thus $f_{21} \in L_1^+$.
Next, we shall define the following functionals

$$f_0,\, f_1,\, f_2;\, E \longrightarrow I\!R,$$

by

$$\begin{aligned} f_0(x,\,u) &:= \lambda_0 \int_0^T [\Phi_x(x^\circ(t),\,u^\circ(t),\,t)x(t) + \Phi_u(x^\circ(t),\,u^\circ(t),\,t)u(t)]dt \\ f_2(x,\,u) &:= f_{21}(x,\,u) + \langle a,\,x(T)\rangle = f_{21}(x,\,u) + f_{22}(x,\,u). \end{aligned}$$

Then $f_0 \in K_0^+,\ f_1 \in K_1^+,\ f_2 \in K_2^+$, and also

$$f_0 + f_1 + f_2 = 0,$$

not all these functionals are zero, because by hypothesis λ_0 and ψ are not both zero. From the convexity conditions, it follows the global-minimality of $(x^\circ,\,u^\circ)$. □

5. Modification of Boundary Conditions

We now discuss problem 1.1 with modified boundary condition. We replace the end condition of (1.4) by a more general condition, in other word, we consider the following optimal control problem

Problem 5.1.

$$\int_0^T \Phi(x(t),\,u(t),\,t)dt \longrightarrow \min \text{ loc.} \tag{5.1}$$

$$(x,\,u) \in E := \mathcal{PW}([0,T];\mathbb{R}^n) \times L_\infty^r([0,T];\mathbb{R}^r), \tag{5.2}$$

$$\dot{x}(t) = \varphi(x(t),\,u(t),\,t), \qquad x(0) = x_0 \tag{5.3}$$

$$x_0 \in I\!R^n;\ G_i(x(T)) = 0, \quad i = 1,2,\ldots,q. \tag{5.4}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1,2,3,\ldots,p. \tag{5.5}$$

$$u(t) \in M, \quad t \in [0,\,T], \quad a.e., \tag{5.6}$$

where $G_i(x)$ are differentiable scalar functions on $I\!R^n$. So arguing exactly as in the previous problem 1.1, under certain conditions that we will present immediately, the cone of tangent vectors K_2 is the set of points $(\overline{x},\overline{u}) \in E$ such that

$$\begin{aligned} \dot{\overline{x}}(t) &= \varphi_x(x^\circ(t),\,u^\circ(t),\,t)\overline{x}(t) + \varphi_u(x^\circ(t),\,u^\circ(t),\,t)u(t), \quad t \neq t_k && (5.7) \\ \overline{x}(t_k^+) &= \mathcal{J}_k'(x^0(t_k^-))\overline{x}(t_k^-), \quad k = 1,2,3,\ldots,p. && (5.8) \\ & \langle G_i'(x^0(T)),\overline{x}(T)\rangle = 0, \quad i = 1,2,3,\ldots,q. && (5.9) \end{aligned}$$

But, in order to compute the tangent cone K_2 we have to assume the following condition on $G_i'(x^\circ(T))$. Consider the jacobian matrix of

$$G(x) = (G_1(x), G_2(x), G_3(x), \cdots, G_q(x)) \tag{5.10}$$

around the point $x^\circ(T)$

$$\Xi = G'(x^\circ(T)) = \begin{pmatrix} G_{11}'(x^\circ(T)) & G_{12}'(x^\circ(T)) & \cdots & G_{1n}'(x^\circ(T)) \\ G_{21}'(x^\circ(T)) & G_{22}'(x^\circ(T)) & \cdots & G_{2n}'(x^\circ(T)) \\ \vdots & \vdots & \vdots & \vdots \\ G_{q1}'(x^\circ(T)) & G_{q2}'(x^\circ(T)) & \cdots & G_{qn}'(x^\circ(T)) \end{pmatrix}. \tag{5.11}$$

Additional Hypothesis

H) Rank$(\Xi) = q$.

Remark 5.1. *Condition H) is equivalent to say that the operator $\Xi : I\!R^n \to I\!R^q$ is onto (Range$(\Xi) = I\!R^q$), which is equivalent that $(\Xi\Xi^*)^{-1}$ exists. Therefore $\Xi^+ = \Xi^*(\Xi\Xi^*)^{-1}$ is a right inverse of Ξ. So, the equation $\Xi x(T) = \hat{a}$ admits the solution $x(T) = \Xi^*(\Xi\Xi^*)^{-1}\hat{a}$.*

In order to compute the tangent cone, we have to modify the operator P defined in problem 1.1, Let us find the tangent cone to Q_2 at the point $(x^\circ,\, u^\circ)$

$$K_2 := K_T(Q_2,\, (x^\circ,\, u^\circ)).$$

Consider the space $E_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times I\!R^{n(1+p)} \times I\!R^q = E_2$ and the operator: $P : E_1 \to E_2$ defined by

$$P(x,u)(t) = \left(x(t) - x_0 - \int_0^t \varphi(x(l), u(l), l)dl, \quad S(x,u), \quad G(x(T)) \right),$$

where

$$S(x,u) = \left(x(t_1) - \mathcal{J}_1(x(t_1^-)), x(t_2) - \mathcal{J}_2(x(t_2^-)) \cdots, x(t_p) - \mathcal{J}_p(x(t_p^-)) \right),$$

and G is given by (5.10). Then

$$P'(x^0,u^0)(\overline{x},\overline{u}) =$$
$$\left(\overline{x}(t) - \int_0^t (\varphi_x(x^\circ(l),\, u^\circ(l),\, l)\overline{x}(l) + \varphi_u(x^\circ(l),\, u^\circ(l),\, l)\overline{u}(l))dl, \quad S'(\overline{x},\overline{u}), \quad \Xi\overline{x}(T) \right)$$

where

$$S'(\overline{x},\overline{u}) = \left(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-) \right)$$

and Ξ is given by (5.11). We want to find conditions under which the operator $P'(x^0,u^0)$ is onto in order to apply Lustenik Theorem 2.27. So, for $(a(\cdot),b_1,b_2,\ldots,b_p,\hat{a}) \in E_2$, we want to solve the equation

$$P'(x^0,u^0)(\overline{x},\overline{u}) = (a(\cdot),b_1,b_2,\ldots,b_p,\hat{a}).$$

Now, suppose that $\overline{u}=0$. Then, from ([25], pg 89), we know that the following Volterra integral equation

$$z(t) = a(t) + \int_0^t (\varphi_x(x^\circ(l),\, u^\circ(l),\, l)z(l)dl,$$

has a solution $z \in \mathcal{PW}([0,T];\mathbb{R}^n)$.
Next, since the impulsive linear variational equation (3.2) is controllable, for a point $(\bar{b}_1,\bar{b}_2,\ldots,\bar{b}_p) \in I\!R^{np}$ such that

$$\bar{b}_k = b_k - z(t_k) + \mathcal{J}_k'(x^0(t_k^-))z(t_k^-), \quad k=1,2,3,\ldots,p.$$

Then, there exists a control $\overline{u} \in L_\infty^r$ such that the corresponding solution $y(t)$ of (3.2) satisfies

$$y(T) = \Xi^*(\Xi\Xi^*)^{-1}\hat{a} - z(T).$$

Let us make the following change of variable $\overline{x} = y+z$. Then

$$\begin{aligned}
&P'(x^0,u^0)(\overline{x},\overline{u})(t) = \Big((y+z)(t)- \\
&\int_0^t (\varphi_x(x^\circ,u^\circ,l)(y+z)(l) + \varphi_u(x^\circ,u^\circ,l)\overline{u}(l))dl, \quad S'(\overline{x},\overline{u}), \quad \Xi(y+z)(T)\Big) \\
&= \Big(y(t)+a(t)- \\
&\int_0^t (\varphi_x(x^\circ,\, u^\circ,\, l)y(l) + \varphi_u(x^\circ,\, u^\circ,l)\overline{u}(l))dl, \quad S'(\overline{x},\overline{u}), \quad \Xi\Xi^*(\Xi\Xi^*)^{-1}\hat{a}\Big) \\
&= \left(a(t), \quad S'(\overline{x},\overline{u}), \hat{a}\right).
\end{aligned}$$

Now, we shall see that $S'(\overline{x},\overline{u}) = (b_1,b_2,\ldots,b_p)$. In fact,

$$\begin{aligned}
&S'(\overline{x},\overline{u}) = \\
&\big(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-),\cdots\cdots,\overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-)\big) = \\
&\big((y+z)(t_1) - \mathcal{J}_1'(x^0(t_1^-))(y+z)(t_1^-),\cdots,(y+z)(t_p) - \mathcal{J}_p'(x^0(t_p^-))(y+z)(t_p^-)\big) \\
&= \big(\bar{b}_1 + z(t_1) - \mathcal{J}_1'(x^0(t_1^-))z(t_1^-),\cdots\cdots,\bar{b}_p + z(t_p) - \mathcal{J}_p'(x^0(t_p^-))z(t_p^-)\big) \\
&= \left(b_1,b_2,\ldots,b_p\right).
\end{aligned}$$

Therefore, the operator $P'(x^0,u^0)$ is onto. Then, applying Lusternik's Theorem 2.27, we get that tangent cone K_2 is given by

$$K_2 = \{(x,\, u) \in E_1 \,/\, P'(x^\circ,\, u^\circ)(\overline{x},\overline{u}) = 0\}.$$

i.e., K_2 is the set of points $(\overline{x},\overline{u}) \in E_1$ such that

$$\begin{aligned}
\dot{\overline{x}}(t) &= \varphi_x(x^\circ(t),\, u^\circ(t),\, t)\overline{x}(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), t \neq t_k & (5.12)\\
\overline{x}(t_k^+) &= \mathcal{J}_k'(x^0(t_k^-))\overline{x}(t_k^-), \quad k=1,2,3,\ldots,p. & (5.13)\\
\Xi\overline{x}(T) &= 0. & (5.14)
\end{aligned}$$

Consider the following linear subspaces

$$L_1 = \{(\overline{x}, \overline{u}) \in E_1/(5.12) - (5.13) \quad \text{hold}\}, \quad L_2 = \{(\overline{x}, \overline{u}) \in E_1/ \quad \Xi\overline{x}(T) = 0\}.$$

Then, $K_2 = L_1 \cap L_2$. Now, let us compute K_2^+. By Proposition 2.40, we have that $f_{22} \in L_2^+$ if, and only if, there exists $a \in I\!R^k$ such that

$$f_{22}(x,\, u) = \langle a,\, \Xi x(T)\rangle \qquad ((x,\, u) \in E).$$

Moreover, the controllability of systems (3.1)- (3.2) implies that $L_1 + L_2$ is closed, then it follows that $L_1^+ + L_2^+$ is w^*- closed; hence by Lemma 2.5 we obtain that

$$K_2^+ = L_1^+ + L_2^+.$$

Since L_1 is a linear subspace, it follows from Theorem 10.1 of (See [18, pg 59]) that,

for any $f_{21} \in L_1^*$, $f_{21}(\overline{x}, \overline{u}) = 0$ for all $(\overline{x}, \overline{u})$ satisfying (5.12)-(5.13).

Euler-Lagrange Equation.

Clearly that K_0, K_1, K_2, are convex cones. Hence, by Theorem 2.15 there are functionals $f_i \in K_i^+ \quad (i = 0,\, 1,\, 2,)$ not all zero such that

$$f_0 + f_1 + f_2 = f_0 + f_1 + f_{21} + f_{22} = 0. \tag{5.15}$$

Equation (5.15) takes the following form

$$\begin{cases} -\lambda_0 \displaystyle\int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt+ \\ \\ +f_1'(x,\, u) + f_{21}(x,\, u) + \langle a,\, \Xi x(T)\rangle = 0, \quad ((x,\, u) \in E). \end{cases} \tag{5.16}$$

Now, for all $u \in L_\infty^r$ there exists x, solution of equation (3.2) with $x(0) = 0$, then $(x,\, u) \in L_1$. Therefore $f_{21}(x,\, u) = 0$.
Let ψ be a solution of the system

$$\begin{cases} \dot{\psi}(\tau) = -\varphi_x^*(x^\circ(\tau),\, u^\circ(\tau),\, \tau)\psi(\tau) + \lambda_0\Phi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau) \\ \psi(T) = \Xi^* a \end{cases}$$

Multiplying both sides of this equation by $\overline{x}$ and integrating by parts from 0 to T, we get

$$
\begin{aligned}
&\lambda_0 \int_0^T \Phi_x(x^\circ,\, u^\circ,\, t)\overline{x}(t)dt - \langle a,\, \Xi\overline{x}(T)\rangle = \\
&\int_0^T \langle \dot{\psi}(t),\, \overline{x}(t)\rangle dt + \int_0^T \langle \varphi_x^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{x}(t)\rangle dt - \langle a,\, \Xi\overline{x}(T)\rangle = \\
&\langle \psi(t),\, \overline{x}(t)\rangle]_0^T - \int_0^T \langle \psi(t),\, \dot{\overline{x}}(t)\rangle dt + \int_0^T \langle \varphi_x^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{x}(t)\rangle dt - \langle a,\, E\overline{x}(T)\rangle = \\
&\langle \Xi^* a,\, \overline{x}(T)\rangle - \langle \psi(0),\, \overline{x}(0)\rangle - \langle a,\, \Xi\overline{x}(T)\rangle + \int_0^T \langle \psi(t), \varphi_x(x^\circ,\, u^\circ,\, t)\overline{x}(t) - \dot{\overline{x}}(t)\rangle dt = \\
&-\int_0^T \langle \psi(t), \varphi_u(x^\circ,\, u^\circ,\, t)\overline{u}(t)\rangle dt = -\int_0^T \langle \varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t), \overline{u}(t)\rangle dt.
\end{aligned}
$$

Then from Euler–Lagrange equation (5.15), we obtain for ($u \in L_\infty^r[0,\, T]$), that

$$
f_1'(u) = \int_0^T \langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)\rangle dt. \tag{5.17}
$$

Since f_1' is a support of Q_1' at the point $u^\circ \in Q_1'$, from example 2.44, it follows that

$$
\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, U - u^\circ(t)\rangle \geq 0,
$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

□

Remark 5.2. *Now, we will see that under these assumptions, the case $\lambda_0 = 0,\ \psi = 0$, can not occurs. If $\psi = 0$, then $\psi(T) = \Xi^* a = 0$. Thus*

$$
f_{22}(x,\, u) = \langle a,\, \Xi x(T)\rangle = 0 \qquad ((x,\, u) \in E),
$$

that is $f_{22} \equiv 0$. So, from equation (5.16), and the fact that $\lambda_0 = 0$, which implies that $f_0 = 0$. Also, from (5.17), we have that $f_1'(u) = 0 \quad (u \in L_\infty^r[0,\, T])$; then from Euler–Lagrange Equation it follows that $f_{21} = 0$, hence

$$
f_2 = f_{21} + f_{22} = 0,
$$

which contradicts Theorem 2.15.

Remark 5.3. **Analysis of the exceptional cases.** *In the course of the proof we have to made two additional assumptions: Firstly, we assumed that $K_0 \neq \emptyset$, and secondly, we assumed that system*

$$
\dot{x}(t) = \varphi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \quad t \in (0, \tau], \tag{5.18}
$$

is controllable.

Now, we will prove, that these assumptions are superfluous. In fact, if $K_0 = \emptyset$, then by definition of K_0, we have that

$$\int_0^T [\Phi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)]dt = 0 \quad ((x,\, u) \in E).$$

Let us put $\lambda_0 = 1,\ \psi(T) = \Xi^* a = 0$, then, from last computation, we have that

$$\int_0^T \langle \Phi_x^*(x^\circ,\, u^\circ,\, t)\psi(t), x(t)\rangle dt = -\int_0^T \langle \varphi_u^\star(x^\circ,\, u^\circ,\, t)\psi(t), u(t)\rangle dt,$$

for all $(x,\, u)$ such that x is a solution of equation the (5.18). Then

$$\int_0^T \langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)\rangle dt = 0, \quad (u \in L_\infty^r[0,\, T]),$$

which implies that

$$\langle -\varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t) + \Phi_u(x^\circ,\, u^\circ,\, t),\, U - u^\circ(t)\rangle = 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Remark 5.4. *The controllability of the linear system,*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \in (0, \tau], \tag{5.19}$$

where $A(t) = \varphi_x(x^\circ(t),\, u^\circ(t),\, t)$ and $B(t) = \varphi_u(x^\circ(t),\, u^\circ(t),\, t)$, is equivalent to:

$$B^*(t)[\Psi^*]^{-1}(t)z = 0, \quad \forall t \in [0, T] \Rightarrow z = 0.$$

Here $\Psi(t)$ is the fundamental matrix of the uncontrolled system $\dot{z} = A(t)z$ and $\psi(t) = [\Psi^*]^{-1}(t)z_0$ is a solution of the adjoint initial value problem

$$\dot{z} = -A^*(t)z, \quad z(0) = z_0.$$

Now, suppose that system (5.19) is not controllable, then there is a non-trivial function $\psi \in \mathcal{PW}([0, T]; \mathbb{R}^n)$ that is a solution of

$$\dot{\psi}(t) = -\varphi_x^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t),$$

such that, for almost all $t \in [0,\, T]$ it follows that

$$-\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) = 0.$$

By taking $\lambda_0 = 0$, we get that ψ is a solution of (3.4), and therefore

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t),\, U - u^\circ(t)\rangle \geq 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Throughout this reasoning, we have proved the following theorem:

Theorem 5.1. *Under conditions of Theorem 3.1. Let assume that Rank*$(\Xi) = q$ *and* $(x^\circ,\, u^\circ) \in E$ *be a solution of Problem 5.1:*
Then, there exists $\lambda_0 \in I\!R_{+0}$ *and a function* $\psi \in \mathcal{PW}([0,T];\mathbb{R}^n)$ *such that* λ_0 *and* ψ *both are different from zero, and* ψ *is a solution of the following differential equation*

$$\begin{cases} \dot{\psi}(t) = -\varphi_x^*(x^\circ(\tau),\, u^\circ(\tau),\, \tau)\psi(\tau) + \lambda_0 \Phi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau) \\ \psi(T) = \Xi^* a. \end{cases} \tag{5.20}$$

Moreover, for all $U \in M$ *and almost all* $t \in [0,\, T]$ *the following inequality holds*

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, U - u^\circ(t)\rangle \geq 0. \tag{5.21}$$

Remark 5.5. *Consider the function*

$$H(x,u,\psi,t) = \langle \varphi^*(x,\, u,\, t), \psi(t)\rangle - \lambda_0 \Phi(x,u,t).$$

Then

$$H_u(x^\circ, u^\circ, \psi, t) = \varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t) - \lambda_0 \Phi_u(x^\circ, u^\circ, t).$$

Since a necessary condition for $H(x^\circ, u, \psi, t)$ *to have a maximum on* M*, as a function of* u*, is that* $-H_u(x^\circ, u^\circ, \psi, t)$ *be a support ot* M *at the point* $u^\circ(t)$*, it follows that (3.5) may be paraphrased as follows. If* (x°, u°) *is a solution of Problem (1.1) and the assumptions of Theorem 3.1 hold, then* $H(x^\circ, u, \psi, t)$ *as a function of* u *on* M*, satisfies the necessary conditions for a maximum for almost all* $0 \leq t \leq T$ *at the point* $u = u^\circ(t)$*. A comparison of this statement with the classic maximum principle justifies the designation "local maximum principle". Specifically we have the following:*

$$\begin{aligned} \langle -H_u(x^\circ, u^\circ, \psi, t), U - u^\circ(t)\rangle &\geq 0 \iff \\ H_u(x^\circ, u^\circ, \psi, t)u^\circ(t) &\geq H_u(x^\circ, u^\circ, \psi, t)U. \end{aligned}$$

Hence,

$$H_u(x^\circ, u^\circ, \psi, t)u^\circ(t) = \max_{U \in M} H_u(x^\circ, u^\circ, \psi, t)U, \quad t \in [0,T], \quad a.e.$$

Since the linear system (3.1) is controllable, then slight modification of the proof of Theorem 3.1 allows us to assume that $\lambda_0 = 1$*.*

6. Example

Now, we shall give an example as an applications of the main result of this work. In this regard, we will give below two previous propositions.

Proposition 6.1. *Let* $x_0 \in I\!R_+^n$ *and* $A = (a_{ij})_{n\times n}$ *be a real matrix, such that* $a_{ij} > 0 \quad (i \neq j,\, i,j = 1,2,\ldots,n)$*. Then*

$$e^{At}\, x_0 \in I\!R_+^n, \quad (t \in I\!R).$$

The proof of above proposition is trivial.
Let $M \subset I\!R^r$ be a set, then we define the set Q_M as follows:

$$Q_M := \{u \in L^r_\infty\,[0,\,T]\,/\,u(t) \in M,\ t \in [0,\,T], \quad a.e.\}.$$

Proposition 6.2. *Let $x_0 \in I\!R^n_+$, and $B = (b_{ij})_{n\times r}$ a real matrix. Then there exists $M \subset I\!R^r$ convex and closed, with $\overset{\circ}{M} \neq \emptyset$ such that*

$$\left(e^{At}\,x_0 + \int_0^t e^{A(t-s)} B\,u(s)ds\right) \in I\!R^n_+, \qquad (u \in Q_M,\ t \in [0,\,T],\, a.e.).$$

Proof. Let $\{e_1,\,e_2,\ldots,\,e_n\}$ be the canonical basis of $I\!R^n$, and define

$$\alpha_i \ := \ \min_{t\in[0,\,T]} \langle e_i,\, e^{At}\,x_0\rangle, \quad (i = 1,\,2,\ldots,\,n),$$

$$V \ := \ (\alpha_1,\,\alpha_2,\ldots,\,\alpha_n).$$

Then, by proposition 6.1 it follows that $V \in I\!R^n_+$.
Let $\delta := \min\{\alpha_i\,/\,i = 1,\,2,\ldots,\,n\}$; then for all $x \in I\!R^n$ such that $|x| < \delta$, we have that $V + x \in I\!R^n_+$.
Let us consider

$$K_1 := \max_{t\in[0,\,T]} \|e^{At}\|, \quad K_2 := \max_{t\in[0,\,T]} \|e^{-At}\|.$$

Then

$$\left|\int_0^T e^{A(t-s)}\,B\,u(s)ds\right| < T\,K_1\,K_2\|B\|\|u\|_\infty,$$

and taking

$$M := \left\{U \in I\!R^r\,/\,|U| \le \frac{\delta}{TK_1K_2\|B\|}\right\},$$

we finish the proof. □

Next, we shall consider the following example where Theorem 3.1 can be applied:

Example 6.3. Let $n = 2,\ r = 1$ and suppose that Φ satisfies the same conditions as in the Problem 1.1, furthermore let us consider

$$B \ = \ \begin{pmatrix} b_{11} \\ b_{12} \end{pmatrix}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad a_{12} > 0, \quad a_{21} > 0$$

$$M \ := \ \left\{U \in R\,/\,|U| \le \frac{\delta}{TK_1K_2\|B\|}\right\},$$

where $\delta,\,K_1,\,K_2$ are defined as in Proposition 6.2.
Let us consider the following problem

$$\int_0^T \Phi(x(t),\,u(t),\,t)dt \longrightarrow \min -loc. \tag{6.1}$$

$$(x,\,u)\in\mathcal{PW}([0,T];\mathbb{R}^n)\times L^r_\infty([0,T];\mathbb{R}^r)$$

$$\dot{x}(t)=Ax(t)+Bu(t) \tag{6.2}$$

$$x(0)=x_0,\ x(T)=x_1;\quad x_0,\,x_1\in I\!R^2_+. \tag{6.3}$$

$$x(t_k^+)=x(t_k^-)+\mathcal{J}_k(x(t_k)),\quad k=1,2,3,\dots,p. \tag{6.4}$$

$$u(t)\in M,\,t\in[0,\,T],\quad a.e. \tag{6.5}$$

Let $(x^\circ,\,u^\circ)\in E=\mathcal{PW}([0,T];\mathbb{R}^n)\times L^r_\infty([0,T];\mathbb{R}^r)$ be a solution of the above problem, then conditions of Theorem 3.1 are fulfilled. In fact, clearly condition a) is satisfied. Also, M closed and convex set with $M^\circ\neq\emptyset$.
c) The linear system (6.2) is controllable. Since this is an autonomous system, we assume that Kalman's Rank condition is satisfied (see [12, 13, 27]). i.e.,

$$\text{Rank }[B\vdots AB]=2.$$

d) The linear system (6.2) with impulses (6.4) is controllable if the following condition is assumed:

$$p\max\|\mathcal{J}_k\|<1,\quad k=1,2,\dots,p.$$

(see [8, 29, 31, 33]). Hence, there exist $\lambda_0\in I\!R_+$, $a\in I\!R^2$, and a function $\psi\in C([0,\,T],I\!R^2)$, which is a solution of the equation

$$\dot{\psi}(t)=-A^*(t)\psi(t)+\lambda_0\Phi_x(x^\circ(t),\,u^\circ(t),\,t), \tag{6.6}$$

such that λ_0 and ψ are not both zero, and for all $U\in M$ and almost all $t\in[0,\,T]$, we have that

$$\langle -B^*\,\psi(t)+\lambda_0\,\Phi_u(x^\circ(t),\,u^\circ(t),\,t),\,U-u^\circ(t)\rangle\geq 0$$

or equivalently

$$\max_{U\in M}\langle B^*\,\psi(t)-\lambda_0\,\Phi_u(x^\circ(t),\,u^\circ(t),\,t),\,U\rangle=\langle B^*\,\psi(t)-\lambda_0\,\Phi_u(x^\circ(t),\,u^\circ(t),\,t),\,u^\circ(t)\rangle \tag{6.7}$$

for almost all $t\in[0,\,T]$.
Let us consider the particular case, in which

$$\Phi(x,\,u)=Cu\qquad((x,\,u,\,t)\in I\!R^2\times I\!R\times[0,\,T]),$$

and let us see how should be the controls $u\in L_\infty[0,\,T]$ that solve the problem:

$$\begin{aligned}\dot{\psi}(t)&=-A^*(t)\psi(t)+\lambda_0\Phi_x(x^\circ(t),\,u^\circ(t),\,t),\\ \max(B^*\psi(t)-\lambda_0\,C)U&=(B^*\psi(t)-\lambda_0\,C)u^\circ(t),\quad U\in[-\rho,\,\rho]\end{aligned}$$

for almost all $t\in[0,\,T]$, where $\rho=\delta/K_1\,K_2\,\|B\|T$.

Let

$$N_{B^*} := \{x \in I\!R^2 \,/\, B^* x - \lambda_0\, C = 0\},$$

$$S := \{t \in [0,\, T] \,/\, \psi(t) \notin N_{B^*}\},$$

then $u^{\circ}(t) := \rho \text{ sig } (B^*\psi(t) - \lambda_0\, C)$ if $t \in S$.

This means that the optimal control should be of the "bang–bang" type over the set S.

7. Optimal Control Problem for Impulsive Neutral Differential Equations

In this section we will show how Dubovitskii–Milyutin theory can be applied to generalize the Maximum Principle of [18] to the case of optimal control problems governed by impulsive nonlinear neutral differential equations. We will also see that in a linear dynamics case, under certain additional conditions, the Maximum Principle is a sufficient condition for optimality.

7.1. Maximum Principle for Neutral Differential Equations in the Space $\mathcal{PW}([0,T];\mathbb{R}^n) \times L^r_\infty$.

Let n, $r \in \mathbb{N}$ and $T \in I\!R_+$, and consider the functions Φ, $\varphi, \mathcal{J}_k$:

$$\varphi \;:\; I\!R^n \times I\!R^r \times [0,\, T] \longrightarrow I\!R,$$

$$\Phi \;:\; I\!R^n \times I\!R^r \times [0,\, T] \longrightarrow I\!R^n,$$

$$\begin{array}{rcl} \mathcal{J}_k & : & I\!R^n \longrightarrow I\!R^n, \\ f & : & I\!R^n \longrightarrow I\!R^n \end{array}$$

where $\mathcal{PW}([0,T];\mathbb{R}^n)$ and L^r_∞ are define by

$$\begin{array}{rcl} \mathcal{PW}([0,T];\mathbb{R}^n) & = & \{z : [0,T] \to \mathbb{R}^n : z \in C(J';\mathbb{R}^n), \exists z(t_k^-), z(t_k^-) \\ & & \text{and} \quad z(t_k) = z(t_k^-), \quad k = 1, 2, \dots, p\}, \end{array}$$

where $J = [0,T]$ and $J' = J \backslash \{t_1, t_2, \dots, t_p\}$, endowed with the norm

$$\|z\|_0 = \sup_{t \in [0,T]} \|z(t)\|_{\mathbb{R}^n},$$

and $L^r_\infty = L^r_\infty([0,T];\mathbb{R}^r)$ is the space of measurable function essentially bounded with essential norm.

Let us suppose the following conditions are fulfilled

a) Φ, φ, f and $\mathcal{J}_k$ are continuous functions, with derivatives Φ_x, Φ_u, φ_x, φ_u, $\mathcal{J}_k'$, f' are bounded functions on compact sets of $I\!R^n \times I\!R^r \times [0,\, T]$.

b) $M \subset I\!R^r$ is convex and closed with $\overset{\circ}{M} \neq \emptyset$.

c) The following linear neutral system is controllable on $[0, T]$,

$$\frac{d}{dt}[x(t)+f'(x^\circ(t))x(t)] = \varphi_x(x^\circ(t),\, u^\circ(t),\, t)x(t)+\varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t). \tag{7.1}$$

d) The corresponding impulsive linear variational equations around the point $(x^\circ,\, u^\circ) \in E$ is controllable on $[0, T]$ for any $\bar{b} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_p) \in (I\!R^n)^p$

$$\begin{cases} \dfrac{d}{dt}[(I + f'(x^\circ(t)))x(t)] = \varphi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \\ x(t_k^+) = \mathcal{J}_k'(x^0(t_k^-))x(t_k^-) + \bar{b}_k, \quad k = 1, 2, 3, \dots, p. \end{cases} \tag{7.2}$$

e) The following conditions hold for all $\quad k = 1, 2, 3, \dots, p$

$$\sup_{t\in[0,T]} \|f'(x^\circ(t))\| < 1, \quad f'(x^\circ(t_k))\mathcal{J}_k'(x^0(t_k^-)) = \mathcal{J}_k'(x^0(t_k^-))f'(x^\circ(t_k)). \tag{7.3}$$

Let us consider the following optimal control problem governed by a nonlinear neutral differential equation:

Problem 7.1.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \text{min loc.} \tag{7.4}$$

$$(x,\, u) \in E := \mathcal{PW}([0, T]; \mathbb{R}^n) \times L_\infty^r([0, T]; \mathbb{R}^r), \tag{7.5}$$

$$\frac{d}{dt}[x(t) + f(x(t))] = \varphi(x(t),\, u(t),\, t), \qquad x(0) = x_0 \tag{7.6}$$

$$x(T) = x_1;\ x_1,\, x_0 \in I\!R^n, \tag{7.7}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \dots, p. \tag{7.8}$$

$$u(t) \in M, \quad t \in [0,\, T], \quad a.e., \tag{7.9}$$

Theorem 7.1. *Let us suppose that conditions a) - e) are fulfilled, and* $(x^\circ,\, u^\circ) \in E$ *is a solutions of the Problem 7.1.*
Then, there exists $\lambda_0 \in I\!R_{+0}$ *and a function* $\psi \in \mathcal{PW}([0,T];\mathbb{R}^n)$ *such that* λ_0 *and* ψ *are not both zero.*
Moreover, ψ *is a solution of the following differential equation*

$$\begin{cases} \dot{\psi}(\tau) = -\left(\varphi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau)\Gamma^{-1}(\tau)\right)^* \psi(\tau) + \lambda_0 \Phi_x(x^\circ(\tau),\, u^\circ(\tau),\, \tau) \\ \psi(T) = a \end{cases} \tag{7.10}$$

where $\Gamma(\tau) = I + f'(x^\circ(\tau))$*, and also, for all* $U \in M$ *and almost all* $t \in [0,\, T]$ *it follows*

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, U - u^\circ(t)\rangle \geq 0. \tag{7.11}$$

Proof. Let $\overline{F} : E \longrightarrow I\!R$ be a function defined as follows

$$\overline{F}(x,\, u) = \int_0^T \Phi(x(t),\, u(t),\, t)dt,$$

and let $Q := Q_1 \cap Q_2$ where Q_2, Q_1 are given by pairs sets $(x,\, u) \in E$, which satisfy (7.6)-(7.8) and (7.9) respectively.
Then, Problem 7.1 is equivalent to

$$\begin{cases} \overline{F}(x,\, u) \longrightarrow \min \;\; \text{loc} \\ \\ (x,\, u) \in Q. \end{cases}$$

a) Analysis of the function $\overline{F}$.
   Let $K_0 := K_d(F,\, (x^\circ,\, u^\circ))$ be the decay cone of $\overline{F}$ in the point $(x^\circ,\, u^\circ)$. Then, by Theorem 2.22, we have that

$$K_0 = \{(x,\, u) \in E \,/\, \overline{F}(x^\circ,\, u^\circ)(x,\, u) < 0\}.$$

   Suppose for a moment that $K_0 \neq \emptyset$, then by Theorem 2.29 we obtain

$$K_0^+ = \{-\lambda_0\, \overline{F}(x^\circ,\, u^\circ) \,/\, \lambda_0 \in I\!R_{+0}\}.$$

   By example 2.23, we obtain that

$$\overline{F}'(x^\circ,\, u^\circ)(x,\, u) = \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt, \qquad ((x,\, u) \in E).$$

   Therefore, for all $f_0 \in K_0^+$, there exists $\lambda_0 \in I\!R_{+0}$ such that

$$f_0(x,\, u) = -\lambda_0 \int_0^T [\Phi_x(x^\circ,\, u^\circ,\, t)x(t) + \Phi_u(x^\circ,\, u^\circ,\, t)u(t)]dt, \qquad ((x,\, u) \in E).$$

b) Analysis of constraint Q_1.
Let us consider the set

$$Q_1' := \{u \in L_\infty^r[0, T] \,/\, u(t) \in M, \quad t \in [0, T], \quad a.e.\},$$

and $Q_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times Q_1'$. Moreover, by hypothesis M is convex and closed, with $\overset{\circ}{M} = \emptyset$. So, the following statements hold

i) Q_1, Q_1' are convex and closed.

ii) $\overset{\circ}{Q}_1 \neq \emptyset$, $\overset{\circ}{Q'}_1 \neq \emptyset$.

If we call K_1 the admissible cone to Q_1 in $(x^\circ, u^\circ) \in Q_1$, then

$$K_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times K_1',$$

where K_1' is the admissible cone to Q_1' in $u^\circ \in Q_1'$.

Therefore, for all $f_1 \in K_1^+$ there is $f_1' \in K_1'^+$ such that $f_1 = (0, f_1')$.

By Theorem 2.26, it follows that f_1' is a support of Q_1' at u°.

c) Analysis of the constraint Q_2.

Let us find the tangent cone to Q_2 at the point (x°, u°)

$$K_2 := K_T(Q_2, (x^\circ, u^\circ)).$$

Consider the space $E_1 = \mathcal{PW}([0,T];\mathbb{R}^n) \times I\!R^{n(1+p)} = E_2$ and the operator: $P : E_1 \to E_2$ defined by

$$P(x,u)(t) = (L(x,u)(t), \quad S(x,u), \quad x(T) - x_1),$$

where

$$L(x,u)(t) = x(t) - x_0 - f(x_0) + f(x(t)) - \int_0^t \varphi(x(l), u(l), l)dl,$$

$$S(x,u) = \big(x(t_1) - \mathcal{J}_1(x(t_1^-)), \quad x(t_2) - \mathcal{J}_2(x(t_2^-)) \cdots, \quad x(t_p) - \mathcal{J}_p(x(t_p^-))\big).$$

Then

$$P'(x^0,u^0)(\overline{x},\overline{u}) = (L'(\overline{x},\overline{u}), \quad S'(\overline{x},\overline{u}), \quad \overline{x}(T)),$$

where

$$L'(\overline{x},\overline{u})(t) = \overline{x}(t) + f'(x^\circ(t))\overline{x}(t) - \int_0^t (\varphi_x(x^\circ, u^\circ, l)\overline{x}(l) + \varphi_u(x^\circ, u^\circ, l)\overline{u}(l))dl$$

$$S'(\overline{x},\overline{u}) = \big(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-)\big)\,.$$

We want to find conditions under which the operator $P'(x^0,u^0)$ is onto in order to apply Lustenik theorem 2.27. So, for $(a(\cdot),b_1,b_2,\ldots,b_p,x_1)\in E_2$, we want to solve the equation

$$P'(x^0,u^0)(\overline{x},\overline{u}) = (a(\cdot),b_1,b_2,\ldots,b_p,x_1).$$

Now, suppose that $\overline{u}=0$. Then, we want to solve the following integral differential equation

$$\Gamma(t)z(t) = a(t) + \int_0^t (\varphi_x(x^\circ(l),\,u^\circ(l),\,l)z(l)dl,$$

which is equivalent to the integral equation

$$z(t) = \Gamma^{-1}(t)a(t) + \int_0^t \Gamma^{-1}(t)\varphi_x(x^\circ(l),\,u^\circ(l),\,l)z(l)dl.$$

From ([25], pg 89), we know that this is a Volterra integral equation, which has a solution $z\in\mathcal{PW}([0,T];\mathbb{R}^n)$.
Next, since the impulsive linear variational equation (7.2) is controllable for all points $b\in I\!R^{np}$. In particular, for a point $(\overline{b}_1,\overline{b}_2,\ldots,\overline{b}_p)\in I\!R^{np}$ such that

$$\overline{b}_k = b_k - z(t_k) + \mathcal{J}_k'(x^0(t_k^-))z(t_k^-), \quad k=1,2,3,\ldots,p,$$

there exists a control $\overline{u}\in L^r_\infty$ such that the corresponding solution $y(t)$ of (7.2) satisfies $y(T)=x_1-z(T)$.
Therefore,

$$\Gamma(t)y(t) = \int_0^t (\varphi_x(x^\circ,u^\circ,l)y(l) + \varphi_u(x^\circ,u^\circ,l)\overline{u}(l))dl, \quad t\in[0,T].$$

Let us make the following change of variable $\overline{x}=y+z$. Then

$$\begin{aligned}
&L'(x^\circ,u^\circ)(y+z,\overline{u})(t) = \Gamma(t)y(t) + \Gamma(t)z(t) - \\
&\int_0^t (\varphi_x(x^\circ,u^\circ,l)(y+z)(l) + \varphi_u(x^\circ,u^\circ,l)\overline{u}(l))dl \\
&= \Gamma(t)y(t) + a(t) - \int_0^t (\varphi_x(x^\circ,\,u^\circ,\,l)y(l) + \varphi_u(x^\circ,\,u^\circ,l)\overline{u}(l))dl = a(t).
\end{aligned}$$

Clearly that $x(T)=x_1$. Now, we shall see that $S'(\overline{x},\overline{u})=(b_1,b_2,\ldots,b_p)$. In fact,

$$\begin{aligned}
&S'(\overline{x},\overline{u}) = \\
&\big(\overline{x}(t_1) - \mathcal{J}_1'(x^0(t_1^-))\overline{x}(t_1^-), \cdots\cdots, \overline{x}(t_p) - \mathcal{J}_p'(x^0(t_p^-))\overline{x}(t_p^-)\big) = \\
&\big((y+z)(t_1) - \mathcal{J}_1'(x^0(t_1^-))(y+z)(t_1^-), \cdots, (y+z)(t_p) - \mathcal{J}_p'(x^0(t_p^-))(y+z)(t_p^-)\big) \\
&= \big(\overline{b}_1 + z(t_1) - \mathcal{J}_1'(x^0(t_1^-))z(t_1^-), \cdots\cdots, \overline{b}_p + z(t_p) - \mathcal{J}_p'(x^0(t_p^-))z(t_p^-)\big) \\
&= (b_1,b_2,\ldots,b_p)\,.
\end{aligned}$$

Thus

$$P'(x^0, u^0)(\overline{x}, \overline{u})(t) = (L'(x^\circ, u^\circ)(x, \overline{u}),\, S'(\overline{x}, \overline{u}),\, x(T)) = (a(\cdot), b_1, b_2, \ldots, b_p, x_1).$$

Therefore, the operator $P'(x^0, u^0)$ is onto. Then, applying Lusternik's theorem 2.27, we get that tangent cone K_2 is given by

$$K_2 = \{(x,\, u) \in E_1 \,/\, P'(x^\circ,\, u^\circ)(\overline{x}, \overline{u}) = 0\}.$$

i.e., K_2 is the set of points $(\overline{x}, \overline{u}) \in E_1$ such that

$$\begin{aligned} [\Gamma(t)\overline{x}(t)]' &= \varphi_x(x^\circ(t),\, u^\circ(t),\, t)\overline{x}(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \\ \overline{x}(t_k^+) &= \mathcal{J}_k'(x^0(t_k^-))\overline{x}(t_k^-), \quad k = 1, 2, 3, \ldots, p. \\ \overline{x}(T) &= 0 \end{aligned}$$

From condition (7.3), we can see that this system is equivalent to the following

$$\begin{aligned} [\Gamma(t)\overline{x}(t)]' &= \left(\varphi_x(x^\circ(t),\, u^\circ(t),\, t)\Gamma^{-1}(t)\right)\Gamma(t)\overline{x}(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \\ \Gamma(t_k^+)\overline{x}(t_k^+) &= \mathcal{J}_k'(x^0(t_k^-))\Gamma(t_k^-)\overline{x}(t_k^-), \quad k = 1, 2, 3, \ldots, p. \\ \Gamma(T)\overline{x}(T) &= 0 \end{aligned}$$

Making the change of variable $z(t) = \Gamma(t)\overline{x}(t)$, we get the following equivalent controllable system

$$z(t)' = \left(\varphi_x(x^\circ(t),\, u^\circ(t),\, t)\Gamma^{-1}(t)\right) z(t) + \varphi_u(x^\circ(t),\, u^\circ(t),\, t)u(t), \tag{7.12}$$

$$z(t_k^+) = \mathcal{J}_k'(x^0(t_k^-))z(t_k^-), \quad k = 1, 2, 3, \ldots, p. \tag{7.13}$$

$$z(T) = 0 \tag{7.14}$$

Consider the following linear subspaces

$$L_1 = \{(\overline{z}, \overline{u}) \in E_1/(7.12) - (7.13) \quad \text{hold}\}, \quad L_2 = \{(\overline{z}, \overline{u}) \in E_1/ \quad \overline{z}(T) = 0\}.$$

Then, $K_2 = L_1 \cap L_2$. Now, let us compute K_2^+. By Proposition 2.40, we have that $f_{22} \in L_2^+$ if, and only if, there exists $a \in I\!R^n$ such that

$$f_{22}(x,\, u) = \langle a,\, z(T)\rangle \qquad ((x,\, u) \in E).$$

Moreover, the controllability of systems (7.1)- (7.2) implies that $L_1 + L_2$ is closed, then it follows that $L_1^+ + L_2^+$ is w^*- closed; then by Lemma 2.5, we obtain that

$$K_2^+ = L_1^+ + L_2^+.$$

Since L_1 is a linear subspace, it follows from Theorem 2.28 that, for any

$f_{21} \in L_1^+$, $f_{21}(\overline{z}, \overline{u}) = 0$ for all $(\overline{x}, \overline{u})$ satisfying (7.12)-(7.13).

e) Euler-Lagrange equation.

It is easy to see that K_0, K_1, K_2, are convex cones. Hence, by Theorem 2.15 there are functionals $f_i \in K_i^+$ $(i = 0, 1, 2,)$ not all zero such that

$$f_0 + f_1 + f_2 = f_0 + f_1 + f_{21} + f_{22} = 0. \tag{7.15}$$

Equation (7.15) can be written in the following form

$$-\lambda_0 \int_0^T [\Phi_x(x^\circ, u^\circ, t)x(t) + \Phi_u(x^\circ, u^\circ, t)u(t)]dt +$$
$$f_1'(x, u) + f_{21}(x, u) + \langle a, x(T)\rangle = 0, \quad ((x, u) \in E).$$

Now, for all $\overline{u} \in L_\infty^r$ there exist $\overline{z}$, solution of system (7.12)-(7.13) with $\overline{z}(0) = 0$. Then $(\overline{z}, u) \in L_1$. Therefore $f_{21}(\overline{z}, \overline{u}) = 0$.

Let ψ be the solution of equation (7.10), that is

$$\begin{cases} \dot{\psi}(\tau) = -\left(\varphi_x(x^\circ(\tau), u^\circ(\tau), \tau)\Gamma^{-1}(\tau)\right)^* \psi(\tau) + \lambda_0 \Phi_x(x^\circ(\tau), u^\circ(\tau), \tau) \\ \psi(T) = a \end{cases}$$

Multiplying both sides of this equation by $\overline{z} = \Gamma(\tau)\overline{x}$ and integrating from 0 to T, we get

$$\begin{aligned}
&\lambda_0 \int_0^T \Phi_x(x^\circ, u^\circ, t)\overline{z}(t)dt - \langle a, \overline{z}(T)\rangle = \int_0^T \langle \dot{\psi}(t), \overline{z}(t)\rangle dt \\
&+ \int_0^T \langle \left(\varphi_x(x^\circ(t), u^\circ(t), t)\Gamma^{-1}(\tau)\right)^* \psi(t), \overline{z}(t)\rangle dt - \langle a, \overline{z}(T)\rangle = \\
&\langle \psi(t), \overline{z}(t)\rangle]_0^T - \int_0^T \langle \psi(t), \dot{\overline{z}}(t)\rangle dt \\
&+ \int_0^T \langle \left(\varphi_x(x^\circ(t), u^\circ(t), t)\Gamma^{-1}(\tau)\right)^* \psi(t), \overline{z}(t)\rangle dt - \langle a, \overline{z}(T)\rangle = \\
&\langle \psi(T), \overline{z}(T)\rangle - \langle \psi(0), \overline{z}(0)\rangle - \langle a, \overline{z}(T)\rangle \\
&+ \int_0^T \langle \psi(t), \varphi_x(x^\circ, u^\circ, t)\Gamma^{-1}(\tau)\overline{z}(t) - \dot{\overline{z}}(t)\rangle dt = \\
&\int_0^T \langle \psi(t), \varphi_x(x^\circ, u^\circ, t)\overline{x}(t) - [\Gamma(\tau)\overline{x}(t)]'\rangle dt = \\
&- \int_0^T \langle \psi(t), \varphi_u(x^\circ, u^\circ, t)\overline{u}(t)\rangle dt = - \int_0^T \langle \varphi_u^*(x^\circ, u^\circ, t)\psi(t), \overline{u}(t)\rangle dt.
\end{aligned}$$

Then, from Euler–Lagrange equation (7.15), we obtain for $(\overline{u} \in L_\infty^r[0, T])$, that

$$f_1'(t) = \int_0^T \langle -\varphi_u^*(x^\circ(t), u^\circ(t), t)\psi(t) + \lambda_0 \Phi_u(x^\circ(t), u^\circ(t), t)u(t)\rangle dt. \tag{7.16}$$

Since f'_1 is a support of Q'_1 at the point $u^\circ \in Q'_1$, from example 2.44, it follows that

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \lambda_0\, \Phi_u(x^\circ(t),\, u^\circ(t),\, t),\, U - u^\circ(t)\rangle \geq 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.
Now, we will see that the case $\lambda_0 = 0,\ \psi = 0$, is not possible. In fact

If $\psi = 0$, then $\psi(T) = a = 0$. Thus

$$f_{22}(x,\, u) = \langle a,\, x(T)\rangle = 0 \qquad ((x,\, u) \in E),$$

that is $f_{22} \equiv 0$. So, from the fact that $\lambda_0 = 0$, we get that $f_0 = 0$. Also, from (7.16), we have that $f'_1(u) = 0 \quad (u \in L^r_\infty[0,\, T])$; then from Euler– Lagrange equation it follows that $f_{21} = 0$, where

$$f_2 = f_{21} + f_{22} = 0,$$

which contradicts Theorem 2.15.

So far, we have two additional assumptions:

Firstly, we assumed that $K_0 \neq \emptyset$, and secondly, we assumed that the system

$$[\Gamma(t)x(t)]' = \varphi_x(x^\circ,\, u^\circ,\, t)x(t) + \varphi_u(x^\circ,\, u^\circ,\, ,t)u(t)$$

is controllable.

Now, we will prove, that these assumptions are superfluous. In fact, if $K_0 = \emptyset$, then by definition of K_0, we have that

$$\int_0^T [\Phi_x(x^\circ(t),\, u^\circ(t),\, t)x(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t)u(t)]dt = 0 \quad ((x,\, u) \in E).$$

Let us put $\lambda_0 = 1,\ \psi(T) = a = 0$, then, from last computation, we have that

$$\int_0^T \Phi_x(x^\circ,\, u^\circ,\, t)x(t)dt = -\int_0^T \langle \varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t), u(t)\rangle dt,$$

for all $(x,\, u)$ such that x is a solution of equation the (7.12)-(7.13). Then

$$\int_0^T \langle \varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) + \Phi_u(x^\circ(t),\, u^\circ(t),\, t), u(t)\rangle dt = 0 \quad (u \in L^r_\infty[0,\, T])$$

which implies that

$$\langle -\varphi_u^*(x^\circ,\, u^\circ,\, t)\psi(t) + \Phi_u(x^\circ,\, u^\circ,\, t),\, U - u^\circ(t)\rangle = 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Now, suppose that system (7.1) is not controllable, then there is a non-trivial function $\psi \in \mathcal{PW}([0,T];\mathbb{R}^n)$ that is a solution of

$$\dot{\psi}(t) = (\varphi_x(x^\circ(t),\, u^\circ(t),\, t)\Gamma^{-1}(t))^*\psi(t),$$

such that, for all $t \in [0,\, T]$ it follows that

$$-\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t) = 0.$$

By taking $\lambda_0 = 0$, we get that ψ is a solution of (7.10), and therefore

$$\langle -\varphi_u^*(x^\circ(t),\, u^\circ(t),\, t)\psi(t),\, U - u^\circ(t)\rangle \geq 0,$$

for all $U \in M$ and almost all $t \in [0,\, T]$.

Thus, the proof of Theorem 7.1 is totally completed.

□

8. Open Problems

Our first open problem concerns with optimal control problems for impulsive nonlinear neutral differential equations with modified boundary condition. In other word, we want to propose the following optimal control problem for future research

8.1. Open Problem

Problem 8.1.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min \text{ loc.} \tag{8.1}$$

$$(x,\, u) \in E := \mathcal{PW}([0,T];\mathbb{R}^n) \times L_\infty^r([0,T];\mathbb{R}^r), \tag{8.2}$$

$$\frac{d}{dt}[x(t) + f(x(t))] = \varphi(x(t),\, u(t),\, t), \qquad x(0) = x_0 \tag{8.3}$$

$$x_0 \in I\!R^n;\ \mathbf{G_i(x(T)) = 0}, \quad i = 1, 2, \ldots, q. \tag{8.4}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{8.5}$$

$$u(t) \in M, \quad t \in [0,\, T], \quad a.e., \tag{8.6}$$

8.2. Open Problem

Second open problem is about optimal control problem on time scales. Basically, we want to analyze the following optimal control problem on time scales for our future investigation:

PROBLEM 8.2.

$$\int_0^T \Phi(x(t),\, u(t),\, t)\Delta t \longrightarrow \min \text{ loc.} \tag{8.7}$$

$$(x,\, u) \in E := PC([0,T]_{\mathbb{T}}; I\!R^n) \times C_{rd}([0,\tau]_{\mathbb{T}}, I\!R^r), \tag{8.8}$$

$$x^{\Delta}(t) = \varphi(x(t),\, u(t),\, t), \qquad x(0) = x_0 \tag{8.9}$$

$$x_0 \in I\!R^n;\ \mathbf{G_i(x(T)) = 0}, \quad i = 1, 2, \dots, q. \tag{8.10}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \dots, p. \tag{8.11}$$

$$u(t) \in M, \quad t \in [0,\, T]_{\mathbb{T}}, \quad a.e., \tag{8.12}$$

where the state function $x(t) \in I\!R^n$, the control u belongs to $C_{rd}([0,\tau]_{\mathbb{T}}, I\!R^r)$, the points $t_k \in \mathbb{T}$ are right dense for $k = 1, \dots, p$ with $0 \le t_1 < \cdots < t_p < \tau$, $x(t_k^+) = \lim\limits_{h\to 0^+} x(t_k + h)$, $x(t_k^-) = \lim\limits_{h\to 0^+} x(t_k - h)$ denotes the left and right limits of $x(t)$ at $t = t_k$ in terms of time scales. Also, we consider the Banach space:

$$\begin{aligned} PC([0,T]_{\mathbb{T}}; I\!R^n) &= \{x : [0,\tau]_{\mathbb{T}} \longrightarrow I\!R^n : x \in C(J'; I\!R^n), \text{there exist} \quad x(t_k^+), x(t_k^-) \\ &\quad\ \text{and } x(t_k) = x(t_k^-), k = 1, 2, \dots, p\} \end{aligned}$$

where $J' = [0,T]_{\mathbb{T}} \setminus \{t_1, \dots, t_p\}$, is endowed with the norm

$$\|x\|_{PC} = \sup\{\|x(t)\| : t \in [0,T]_{\mathbb{T}}\}.$$

8.3. Open Problem

In the third problem we will study an optimal control problem governed by differential equations of the neutral type on time scales:

PROBLEM 8.3.

$$\int_0^T \Phi(x(t),\, u(t),\, t)\Delta t \longrightarrow \min \text{ loc.} \tag{8.13}$$

$$(x,\, u) \in E := PC([0,T]_{\mathbb{T}}; I\!R^n) \times C_{rd}([0,\tau]_{\mathbb{T}}, I\!R^r), \tag{8.14}$$

$$[x(t) + f(t, x(t))]^{\Delta} = \varphi(x(t),\, u(t),\, t), \qquad x(0) = x_0 \tag{8.15}$$

$$x_0 \in I\!R^n; \;\; \mathbf{G_i}(\mathbf{x}(\mathbf{T})) = \mathbf{0}, \quad i = 1, 2, \ldots, q. \tag{8.16}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{8.17}$$

$$u(t) \in M, \quad t \in [0,\, T]_{\mathbb{T}}, \quad a.e. \tag{8.18}$$

8.4. Open Problem

Our fourth open problem can be an optimal control system governed by an impulsive equation of the neutral type and nonlocal conditions. It can also be formulated in time scale .

Problem 8.4.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min \text{loc.} \tag{8.19}$$

$$(x,\, u) \in E := \mathcal{PW}([0, T]; \mathbb{R}^n) \times L^r_\infty([0, T]; \mathbb{R}^r), \tag{8.20}$$

$$\frac{d}{dt}[x(t) + f(t, x(t))] = \varphi(x(t),\, u(t),\, t), \qquad \mathbf{x}(\mathbf{0}) = \mathbf{g}(\mathbf{x}) + \mathbf{x_0} \tag{8.21}$$

$$x_0 \in I\!R^n; \;\; \mathbf{G_i}(\mathbf{x}(\mathbf{T})) = \mathbf{0}, \quad i = 1, 2, \ldots, q. \tag{8.22}$$

$$x(t_k^+) = x(t_k^-) + \mathcal{J}_k(x(t_k)), \quad k = 1, 2, 3, \ldots, p. \tag{8.23}$$

$$u(t) \in M, \quad t \in [0,\, T], \quad a.e. \tag{8.24}$$

8.5. Open Problem

Our fifth open problem deals with an optimal control problem for non-autonomous semilinear neutral differential equations with unbounded delay, non-instantaneous impulses, and nonlocal conditions. Specifically, we are interested in finding a maximal principle for the following problem.

Problem 8.5.

$$\int_0^T \Phi(x(t),\, u(t),\, t)dt \longrightarrow \min \text{loc.} \tag{8.25}$$

$$(x,\, u) \in E := \mathcal{PW}((-\infty, T]; \mathbb{R}^n) \times L^r_\infty([0, T]; \mathbb{R}^r), \tag{8.26}$$

$$\frac{d}{dt}[x(t) - g(t, x_t)] = \mathcal{A}(t)x(t) + \mathsf{B}(t)u(t) + f(t, x_t, u(t)), \quad t \in \bigcup_{k=0}^{N} J_k^1, \tag{8.27}$$

$$x(t) = \Gamma_k(t, x(t_k^-), u(t_k^-)), \quad t \in J_k^2, k = 1, \ldots, N, \tag{8.28}$$

$$x(s) + \zeta(x_{\lambda_1}, \ldots, x_{\lambda_q})(s) = \phi(s) \quad s \in (-\infty, 0]. \tag{8.29}$$

$$x(T) = x_1; \;\; x_1 \in I\!R^n, \quad \phi \in \mathfrak{L}, \tag{8.30}$$

$$u(t) \in M, \quad t \in [0, T]_{\mathbb{T}}, \quad a.e., \tag{8.31}$$

where the state function $x(t)$ takes values in $I\!R^n$, meanwhile the control $u(\cdot)$ belongs to $L^r_\infty([0,T];\mathbb{R}^r)$, the space of admissible control functions. The matrices $\mathcal{A}(t)$ and $\mathsf{B}(t)$ are continuous of order $n \times n$ and $n \times m$, separately. The functions $x_t : (-\infty, 0] \longrightarrow I\!R^n$ given by $x_t(\theta) = x(t+\theta), \theta \leq 0$, belong to the phase space $\mathfrak{L}$ and represent the history of x up to t. Here $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_q < T$ are prefixed numbers selected conveniently according the phenomenon to be modelled. Similarly, $s_0 = 0 < t_1 < s_1 < t_2 < \cdots < t_N < s_N < t_{N+1} = T$, $J_0 = [0, t_1], J^1_k = (s_k, t_{k+1}]$ and $J^2_k = (t_k, s_k]$. The functions $g : [0,T] \times \mathfrak{L} \to I\!R^n$, $f : [0,T] \times \mathfrak{L} \times I\!R^m \to I\!R^n$, $x_t \in \mathfrak{L}$, $\phi \in \mathfrak{L}$, $\Gamma_k : (t_k, s_k] \times I\!R^n \times I\!R^m \to I\!R^n$ and $\zeta : \mathfrak{L}^q \to \mathfrak{L}$ are appropiate functions. In particular, $\Gamma_k, k = 1, 2, ...$, describes the non-instantaneous impulses in the model and ζ denotes the nonlocal conditions. For more information about the controllability of differential equations with noninstantaneous pulses, nonlocal conditions, and infinite delay, one can review the following references [11, 17, 28, 30].

9. Conclusion and Final Remark

As we have seen in this work, Pontryaguin's maximum principle is still valid for optimal control problems governed by differential equations with impulses as long as the impulses are small in some sense; that is, the linear variational equation around the optimal point is controllable. In many articles, of which we can mention ([7, 8, 10, 28, 29, 30, 31, 32, 33, 36]), it has already been verified that the controllability of the linear system is robust if we add impulses to the differential equation, delays and the non-local conditions as disturbances of the system. So, here we have seen that the maximum principle remains invariant under certain conditions on the impulses, so we believe that we can also say something if we add non-local conditions, and also consider dynamical equations on time scales.

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Hugo Leiva
email: hleivatoka@gmail.com
ORCID: 0000-0002-3521-6253
School of Mathematical and Computational Sciences
Yachay Tech
Ibarra, Imbabura
ECUADOR

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