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Nano b-I-Continuous Functions and Nano b-I-Open Functions

Jamal M. Mustafa

Abstract: The aim of this paper is to define and study certain new classes of continuous, irresolute and open functions namely nano b-I-continuous, nano b-I-irresolute and nano b-I-open functions in nano ideal topological spaces. Some characterizations and properties regarding these concepts are discussed. All these concepts will be helpful for further generalizations of nano continuous mappings in nano ideal topological spaces.

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1. Introduction

Thivagar and Richard [23] established the field of nano topological spaces. In 2016, Thivagar and Devi [21] introduced the notion of nano local functions and explore the field of nano topological spaces. In 2018, Parimala and Jafari [18] introduced the notion of nano I-continuous functions in nano ideal topological spaces. Jamal M. Mustafa [13 - 16] studied weakly nano semi-I-open sets and weakly nano semi-I-continuous functions and some covering properties using the b-open sets. In this paper we introduce and study the new classes of continuous, irresolute and open functions namely $nano\ b-I-continuous$, $nano\ b-I-irresolute$ and $nano\ b-I-open$ functions in nano ideal topological spaces and we discuss some of their properties.

Let (D, ζ) be a topological space and $A \subseteq D$. The complement of A in D, the closure of A, the interior of A and the power set of A will be denoted by $D - A = A^c$, $Cl\,(A)$, $Int\,(A)$ and $\mathcal{P}\,(A)$, respectively. The subject of ideals in topological spaces has been studied by Kuratowski [10] and Vaidyanathaswamy [25]. An ideal on a topological space (D, ζ) is defined as a non-empty collection I of subsets of D satisfying the following two conditions: (1) If $A \in I$ and $B \subseteq A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. An ideal topological space is a topological space (D, ζ) with an ideal I on D and is denoted by (D, ζ, I). For a subset $A \subseteq D$, $A^*(I) = \{x \in D : U \cap A \notin I \text{ for every } U \in \zeta \text{ with } x \in U\}$ is called the local function of A with respect to I and ζ [10]. We simply write A^* instead of $A^*(I)$ in case there is no chance of confusion. It is well known that $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

First we shall recall some definitions used in the sequel.

Definition 1.1. Let A be a subset of a topological space (D, ζ). Then

a) A is called $semi-open$ [9] if $A \subseteq Cl(Int(A))$.

b) A is called $pre-open$ [10] if $A \subseteq Int(Cl(A))$.

c) A is called $\alpha-open$ [10] if $A \subseteq Int(Cl(Int(A)))$.

c) A is called $b-open$ [1] if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$.

d) A is called $semi-closed$ [4] if it is the complement of a semi-open set.

e) The $semi-closure$ of A [4], denoted by $sCl\,(A)$, is the smallest semi-closed set that contains A.

Definition 1.2. A subset A of an ideal topological space (D, ζ, I) is said to be

a) $I-open$ [9] if $A \subseteq Int(A^*)$.

b) $semi-I-open$ [7] if $A \subseteq Cl^*(Int(A))$.

c) $pre-I-open$ [5] if $A \subseteq Int(Cl^*(A))$.

d) $b-I-open$ [6] if $A \subseteq Cl^*(Int(A)) \cup Int(Cl^*(A))$.

2. Preliminaries

Definition 2.1. [23] Let U be a non-empty finite set of all objects called the universe and R be an equivalence relation on U named as indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space. Let $D \subseteq U$. Then,

(1) The lower approximation of D with respect to R is the set of all objects which can be for certain classified as D with respect to R and is denoted by $L_R(D)$. $L_R(D) = \cup\{R(x) : R(x) \subseteq D, x \in U\}$ where $R(x)$ denotes the equivalence class determined by $x \in U$.

(2) The upper approximation of D with respect to R is the set of all objects which can be possibly classified as D with respect to R and is denoted by $U_R(D) = \cup\{R(x) : R(x) \cap D \neq \phi, x \in U\}$.

(3) The boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not$-D$ with respect to R and is denoted by $B_R(D)$. $B_R(D) = U_R(D) - L_R(D)$.

Remark. [23] If (U, R) is an approximation space and $D, E \subseteq U$, then

(1) $L_R(D) \subseteq D \subseteq U_R(D)$.

(2) $L_R(\phi) = U_R(\phi) = \phi$.

(3) $L_R(U) = U_R(U) = U$.

(4) $U_R(D \cup E) = U_R(D) \cup U_R(E)$.

(5) $U_R(D \cap E) \subseteq U_R(D) \cap U_R(E)$.

(6) $L_R(D \cup E) \supseteq L_R(D) \cup L_R(E)$.

(7) $L_R(D \cap E) = L_R(D) \cap L_R(E)$.

(8) $L_R(D) \subseteq L_R(E)$ and $U_R(D) \subseteq U_R(E)$ whenever $D \subseteq E$.

(9) $U_R(D^c) = [L_R(D)]^c$ and $L_R(D^c) = [U_R(D)]^c$.

(10) $U_R(U_R(D)) = L_R(U_R(D)) = U_R(D)$.

(11) $L_R(L_R(D)) = U_R(L_R(D)) = L_R(D)$.

Definition 2.2. [23] Let U be the universe, R be an equivalence relation on U and $\zeta_R(D) = \{U, \phi, L_R(D), U_R(D), B_R(D)\}$ where $D \subseteq U$. Then by the last remark, $\zeta_R(D)$ satisfies the following axioms:

(1) U and $\phi \in \zeta_R(D)$.

(2) The union of the elements of any subcollection of $\zeta_R(D)$ is in $\zeta_R(D)$.

(3) The intersection of the elements of any finite subcollection of $\zeta_R(D)$ is in $\zeta_R(D)$.

Then $\zeta_R(D)$ is a topology on U called the nano topology on U with respect to X. $(U, \zeta_R(D))$ is called the nano topological space. Elements of the nano topology are known as nano open sets in U and the complement of a nano open set is called nano closed.

Definition 2.3. [23] If $\zeta_R(D)$ is the nano topology on U with respect to D, then the set $B = \{U, L_R(D), B_R(D)\}$ is the basis for $\zeta_R(D)$.

Definition 2.4. [23] If $(U, \zeta_R(D))$ is a nano topological space with respect to D where $D \subseteq U$ and if $A \subseteq U$, then

(1) The nano interior of the set A is defined as the union of all nano open subsets contained in A and is denoted by $nInt(A)$. $nInt(A)$ is the largest nano open subset of A.

(2) The nano closure of the set A is defined as the intersection of all nano closed sets containing A and is denoted by $nCl(A)$. $nCl(A)$ is the smallest nano closed set containing A.

Definition 2.5. [23] Let $(U, \zeta_R(D))$ be a nano topological space and $A \subseteq U$. Then A is said to be:

(1) *nano semi − open* if $A \subseteq nCl(nInt(A))$.

(2) *nano pre open* if $A \subseteq nInt(nCl(A))$.

Definition 2.6. Let $(U, \zeta_R(D))$ and $(V, \zeta_{R'}(E))$ be two nano topological spaces. A function $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E))$ is called:

(1) nano continuous [24] if $f^{-1}(B)$ is nano open in U for every nano open set B in V .

(2) nano semi-continuous [20] if $f^{-1}(B)$ is nano semi-open in U for every nano open set B in V .

(3) nano precontinuous [22] if $f^{-1}(B)$ is nano preopen in U for every nano open set B in V .

(4) nano open if $f(A)$ is nano open in V for every nano open set A in U.

(5) nano closed if $f(C)$ is nano closed in V for every nano closed set C in U.

3. Nano ideal topological spaces

In 2016, Thivagar and Devi [21] considered the nano local function in nano ideal topological space and they obtained a new topology. A nano ideal topological space is a nano topological space $(U, \zeta_R(D))$ with an ideal I on U and is denoted by $(U, \zeta_R(D), I)$. For a subset $A \subseteq U$, $nA^*(I) = \{x \in U : W \cap A \notin I$ for every $W \in \zeta_R(D)$ with $x \in W\}$ is called the nano local function of A with respect to I and $\zeta_R(D)$ [21]. We simply write nA^* instead of $nA^*(I)$ in case there is no chance of confusion. It is well known that $nCl^*(A) = A \cup nA^*$ defines a nano closure operator for $(\zeta_R(D))^*(I)$.

Theorem 3.1. *[21] Let $(U, \zeta_R(D))$ be a nano topological space with ideals I, J on U and A, B be subsets of U. Then the following statements are true:*

(i) if $A \subseteq B$, then $nA^* \subseteq nB^*$

(ii) if $I \subseteq J$, then $nA^*(I) \subseteq nA^*(J)$.

(iii) $nA^* = nCl(nA^*) \subseteq nCl(A)$.

(iv) $n(nA^*)^* = nA^*$.

(v) $nA^* \cup nB^* = n(A \cup B)^*$.

(vi) $nA^* - nB^* = n(A - B)^* - nB^* \subseteq n(A - B)^*$.

(vii) *if* $V \in \tau_R(D)$, *then* $V \cap nA^* = V \cap n(V \cap A)^* \subseteq n(V \cap A)^*$.

(viii) *if* $E \in I$, *then* $n(A \cup E)^* = nA^* = n(A - E)^*$.

Theorem 3.2. *[21] The nano closure operator* nCl^* *satisfies the following conditions:*

(i) $A \subseteq nCl^*(A)$.

(ii) $nCl^*(\phi) = \phi$ *and* $nCl^*(U) = U$.

(iii) *if* $A \subseteq B$ *then* $nCl^*(A) \subseteq nCl^*(B)$.

(iv) $nCl^*(A) \cup nCl^*(B) = nCl^*(A \cup B)$.

(v) $nCl^*(nCl^*(A)) = nCl^*(A)$.

Definition 3.3. *A subset* A *of a nano ideal topological space* $(U, \zeta_R(D), I)$ *is said to be*

(1) nano semi $I - open$ *[21] if* $A \subseteq nCl^*(nInt(A))$.

(2) nano pre $I - open$ *[8] if* $A \subseteq nInt(nCl^*(A))$.

(3) nano $\alpha - I - open$ *[21] if* $A \subseteq nInt(nCl^*(nInt(A)))$.

Definition 3.4. A subset $A \subseteq U$ in a nano ideal topological space $(U, \zeta_R(D), I)$ is said to be *nano* $b - I - open$ [19] if $A \subseteq nCl^*(nInt(A)) \cup nInt(nCl^*(A))$.

The family of all nano b-I-open sets of the space $(U, \zeta_R(D), I)$ will be denoted by $NbIO(U, \zeta_R(D))$.

A subset $A \subseteq U$ in a nano ideal topological space $(U, \zeta_R(D), I)$ is said to be *nano* $b - I - closed$ if its complement is *nano* $b - I - open$.

Theorem 3.5. *For a subset of a nano ideal topological space, the following properties hold:*

(a) Every nano semi $I - open$ *set is nano b-I-open.*

(b) Every nano pre $I - open$ *set is nano b-I-open.*

(c) Every nano $\alpha - I - open$ *set nano b-I-open.*

The converse of each part in the above theorem need not be true as shown in the following example.

Example 3.6. Let $U = \{a, b, c, d\}$ be the universe, $D = \{b, d\} \subseteq U$, $U/R = \{\{a\}, \{b\}, \{c, d\}\}$, $\zeta_R(D) = \{\phi, U, \{b\}, \{c, d\}, \{b, c, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$. Then

(1) The set $\{a, b, d\}$ is a *nano b-I-open set* but it is not nano *semi-I-open.*

(2) The set $\{a, b\}$ is a *nano b-I-open set* but it is not nano *pre-I-open and not nano* $\alpha - I - open$.

Lemma 3.7. *[9] Let A and B be subsets of U in a nano ideal topological space $(U, \zeta_R(D), I)$.*

a) If $A \subseteq B$, *then* $A^* \subseteq B^*$.

b) If $V \in \zeta_R(D)$, *then* $V \cap A^* \subseteq (V \cap A)^*$.

c) A^* *is nano closed in* $(U, \zeta_R(D))$.

Theorem 3.8. *Let $(U, \zeta_R(D), I)$ be an ideal topological space and A, B subsets of U.*

1) If $A_\alpha \in NbIO(U, \zeta_R(D))$ *for each* $\alpha \in \Delta$, *then* $\bigcup\{A_\alpha : \alpha \in \Delta\} \in NbIO(U, \zeta_R(D))$.

2) If $A \in NbIO(U, \zeta_R(D))$ *and* $B \in \zeta_R(D)$, *then* $A \cap B \in NbIO(U, \zeta_R(D))$.

Proof. 1) Since $A_\alpha \in NbIO(U, \zeta_R(D))$, we have

$$\begin{aligned}
\bigcup_{\alpha\in\Delta} A_\alpha &\subseteq \bigcup_{\alpha\in\Delta} [nCl^*(nInt(A_\alpha)) \cup nInt(nCl^*(A_\alpha))] \\
&\subseteq \bigcup_{\alpha\in\Delta} \{[(nInt(A_\alpha)) \cup (nInt(A_\alpha))^*] \cup [nInt(A_\alpha \cup A_\alpha^*)]\} \\
&\subseteq [nInt(\bigcup_{\alpha\in\Delta} A_\alpha) \cup (nInt(\bigcup_{\alpha\in\Delta} A_\alpha))^*] \cup [nInt((\bigcup_{\alpha\in\Delta} A_\alpha) \cup (\bigcup_{\alpha\in\Delta} A_\alpha)^*)] \\
&= nCl^*(nInt(\bigcup_{\alpha\in\Delta} A_\alpha)) \cup nInt(nCl^*(\bigcup_{\alpha\in\Delta} A_\alpha)).
\end{aligned}$$

Hence $\bigcup_{\alpha\in\Delta} A_\alpha \in NbIO(U, \zeta_R(D))$.

2) Let $A \in NbIO(U, \zeta_R(D))$ and $B \in \zeta_R(D)$. Then $A \subseteq nCl^*(nInt(A)) \cup nInt(nCl^*(A))$ and so

$$\begin{aligned}
A \cap B &\subseteq [nCl^*(nInt(A)) \cup nInt(nCl^*(A))] \cap B \\
&= [nCl^*(nInt(A)) \cap B] \cup [nInt(nCl^*(A)) \cap B] \\
&= [[nInt(A) \cup (nInt(A))^*] \cap B] \cup [nInt(A \cup A^*) \cap B] \\
&\subseteq [(nInt(A) \cap B) \cup ((nInt(A)) \cap B)^*] \cup [nInt[(A \cap B) \cup (A^* \cap B)]] \\
&\subseteq [(nInt(A) \cap B) \cup (nInt(A) \cap B)^*] \cup [nInt[(A \cap B) \cup (A \cap B)^*]] \\
&\subseteq [(nInt(A \cap B)) \cup (nInt(A \cap B))^*] \cup [nInt[(A \cap B) \cup (A \cap B)^*]] \\
&= nCl^*(nInt(A \cap B)) \cup nInt(nCl^*(A \cap B)).
\end{aligned}$$

This shows that $A \cap B \in NbIO(U, \zeta_R(D))$. □

The following example shows that the finite intersection of nano b-I-open sets need not be nano b-I-open.

Example 3.9. Let $U = \{a, b, c, d\}$ be the universe, $D = \{b, d\} \subseteq U$, $U/R = \{\{a\}, \{b\}, \{c, d\}\}$, $\zeta_R(D) = \{\phi, U, \{b\}, \{c, d\}, \{b, c, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$. Then the sets $A = \{a, b, d\}$ and $B = \{a, c, d\}$ are *nano b-I-open sets* but $A \cap B = \{a, d\}$ is not *nano b-I-open*.

4. Nano b-I-continuous functions

Definition 4.1. [8] A function $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ is said to be *nano* $semi - I - continuous$ (resp. $nano\ pre - I - continuous$, $nano\ \alpha - I - continuous$) if $f^{-1}(B)$ is nano semi I-open (resp. nano pre I -open, nano α-I-open) set in $(U, \zeta_R(D), I)$ for every nano open set B in $(V, \zeta_{R'}(E))$.

Definition 4.2. A function $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ is called $nano\ b - I - continuous$ if the inverse image of each nano open set in $(V, \zeta_{R'}(E))$ is a nano b-I-open set in $(U, \zeta_R(D), I)$.

Remark.

1) *Every nano continuous function is nano b-I-continuous.*
2) *Every nano semi-I-continuous function is nano b-I-continuous.*
3) *Every nano pre-I-continuous function is nano b-I-continuous.*
4) *Every nano α-I-continuous function is nano b-I-continuous.*

The converse in each part of the above remark need not be true as shown in the following three examples.

Example 4.3. Let $U = \{a, b, c, d\}$ be the universe, $D = \{a, d\} \subseteq U$, $U/R = \{\{a, d\}, \{b\}, \{c\}\}$, $\zeta_R(D) = \{\phi, U, \{a, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$ and let $V = \{a, b, c\}$, $Y = \{a, c, d\} \subseteq V$, $V/R' = \{\{b\}, \{a, c\}, \{d\}\}$, $\zeta_{R'}(E)) = \{\phi, V, \{a, c, d\}\}$. Define $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ by $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$. We note that, $f^{-1}(\{a, c, d\}) = \{a, c, d\}$ is a $nano\ b - I - open$ *set but not nano open.* Hence, f is nano b-I-continuous but not nano continuous.

Example 4.4. Let $U = \{a, b, c, d\}$ be the universe, $D = \{b, d\} \subseteq U$, $U/R = \{\{a, d\}, \{b\}, \{c\}\}$, $\zeta_R(D) = \{\phi, U, \{a, d\}, \{b\}, \{a, b, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$ and let $V = \{a, b, c, d\}$, $E = \{a, b, d\} \subseteq V$, $V/R' = \{\{a\}, \{b, d\}, \{c\}\}$, $\zeta_{R'}(E)) = \{\phi, V, \{a, b, d\}\}$.

(1) Define $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ by $f(a) = c$, $f(b) = b$, $f(c) = a$, $f(d) = d$. Then f is nano b-I-continuous but not nano semi I-continuous.

(2) Define $g : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ by $g(a) = a$, $g(b) = c$, $g(c) = b$, $g(d) = d$. Then g is nano b-I-continuous but not nano pre I-continuous.

Example 4.5. Let $U = \{a, b, c, d\}$ be the universe, $D = \{a, b\} \subseteq U$, $U/R = \{\{a\}, \{b, d\}, \{c\}\}$, $\zeta_R(D) = \{\phi, U, \{a\}, \{b, d\}, \{a, b, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$ and let $V = \{a, b, c\}$, $E = \{a\} \subseteq V$, $V/R' = \{\{a\}, \{b, c\}\}$, $\zeta_{R'}(E)) = \{\phi, V, \{a\}\}$. Define $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ by $f(a) = f(b) = f(c) = a$, $f(d) = c$. Then f is *nano* $b - I - $*continuous* but not *nano* $\alpha - I - $*continuous.*

Theorem 4.6. *For a function $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ the following statements are equivalent:*

1) *f is nano b-I-continuous.*

2) *For each $x \in D$ and each $B \in \zeta_{R'}(E)$ with $f(x) \in B$, there exists $A \in NbIO(U, \zeta_R(D))$ with $x \in A$ such that $f(A) \subseteq B$.*

3) *The inverse image of each nano closed set in $(V, \zeta_{R'}(E))$ is nano b-I-closed in $(U, \zeta_R(D), I)$.*

Proof. Straightforward. □

Definition 4.7. Let $A \subseteq U$ in a nano ideal topological space $(U, \zeta_R(D), I)$ and $x \in U$. Then A is called a *nano* $b - I - $*neighborhood* of x, if there exists a nano b-I-open set B containing x such that $B \subseteq A$.

Theorem 4.8. *For a function $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$, the following statements are equivalent:*

1) f is nano b-I-continuous.

2) For each $x \in U$ and each nano open set B in $(V, \zeta_{R'}(E))$ with $f(x) \in B$, $f^{-1}(B)$ is nano b-I-neighborhood of x.

Proof. (1) ⇒ (2). Let $x \in U$ and let B be a nano open set in $(V, \zeta_{R'}(E))$ such that $f(x) \in B$. By Theorem 4.7, there exists a nano b-I-open set A in $(U, \zeta_R(D), I)$ with $x \in A$ such that $f(A) \subseteq B$. So $x \in A \subseteq f^{-1}(B)$. Hence $f^{-1}(B)$ is a nano b-I-neighborhood of x.

(2) ⇒ (1). Let B be a nano open set in $(V, \zeta_{R'}(E))$ and let $f(x) \in B$. Then by assumption, $f^{-1}(B)$ is a nano b-I-neighborhood of x. Thus for each $x \in f^{-1}(B)$ there exists a nano b-I-open set A_x containing x such that $x \in A_x \subseteq f^{-1}(B)$. Hence $f^{-1}(B) = \cup\{A_x : x \in f^{-1}(B)\}$ and so $f^{-1}(B) \in NbIO(U, \zeta_R(D))$. □

Definition 4.9. A function $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ is called nano $b - I - $*irresolute* if $f^{-1}(B)$ is nano b-I-open in $(U, \zeta_R(D), I)$ for every nano b-I-open set B in $(V, \zeta_{R'}(E))$.

Theorem 4.10. *Let $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E), J)$ and $g : (V, \zeta_{R'}(E), J) \to (W, \zeta_{R''}(Z), K)$ then*

1) gof is nano b-I-continuous if f is nano b-I-continuous and g is nano continuous.

2) gof is nano b-I-continuous if f is nano b-I-irresolute and g is nano b-I-continuous.

If $(U, \zeta_R(D), I)$ is a nano ideal topological space and A is a subset of U, we denote by $\zeta_R(D)_{|A}$ the relative nano topology on A and $I_{|A} = \{A \cap B : B \in I\}$ is obviously an ideal on A.

The proofs of the following two lemmas is similar to the proofs of Lemma 3.14 and Lemma 3.15 in [18].

Lemma 4.11. *Let $(U, \zeta_R(D), I)$ be a nano ideal topological space and A, B be subsets of U such that $B \subseteq A$. Then $nB^*(\zeta_R(D)_{|A}, I_{|A}) = nB^*(\zeta_R(D), I) \cap A$.*

Lemma 4.12. *Let $(U, \zeta_R(D), I)$ be a nano ideal topological space, $A \subseteq U$ and $W \in \zeta_R(D)$. Then $nCl^*(A) \cap W = nCl^*_W(A \cap W)$.*

Theorem 4.13. *Let $(U, \zeta_R(D), I)$ be a nano ideal topological space, $A \subseteq W \in \zeta_R(D)$. If $A \in NbIO(U, \zeta_R(D))$ then $A \in NbIO(W, \zeta_R(D)_{|W}, I_{|W})$.*

Proof. Since $W \in \zeta_R(D)$ and $A \in NbIO(U, \zeta_R(D))$, we have

$$\begin{aligned} A = W \cap A &\subseteq W \cap [nCl^*(nInt(A)) \cup nInt(nCl^*(A))] \\ &= [W \cap (nCl^*(nInt(A)))] \cup [W \cap (nInt(nCl^*(A)))] \\ &\subseteq nCl^*(W \cap nInt(A)) \cup (W \cap nInt(nCl^*(A))) \\ &= nCl^*(Int(W \cap A)) \cup nInt(W \cap nCl^*(A)) \\ &= nCl^*(nInt_W(W \cap A)) \cup nInt_W(W \cap nCl^*(A)). \end{aligned}$$

Since $W \in \zeta_R(X) \subseteq \zeta_R(X)^*$, we obtain

$$\begin{aligned} A = W \cap A &\subseteq W \cap [nCl^*(nInt_W(W \cap A)) \cup nInt_W(W \cap nCl^*(A))] \\ &= [W \cap (nCl^*(nInt_W(W \cap A)))] \cup [W \cap (nInt_W(W \cap nCl^*(A)))] \\ &= nCl^*_W(nInt_W(W \cap A)) \cup nInt_W(W \cap nCl^*(A)) \\ &= nCl^*_W(nInt_W(A)) \cup nInt_W(nCl^*_W(A)). \end{aligned}$$

Then $A \in NbIO(W, \zeta_R(X)_{|W}, I_{|W})$. □

Corollary 4.14. *Let $(U, \zeta_R(D), I)$ be a nano ideal topological space, $W \in \zeta_R(D)$ and $A \in NbIO(U, \zeta_R(D))$, then $W \cap A \in NbIO(U, \zeta_{|U}, I_{|U})$.*

Proof. Since $W \in \zeta_R(D)$ and $A \in NbIO(U, \zeta_R(D))$, by Theorem 3.8, $W \cap A \in NbIO(U, \zeta_R(D))$. Since $W \in \zeta_R(D)$, by Theorem 4.14, $W \cap A \in NbIO(W, \zeta_R(D)_{|W}, I_{|W})$. □

Theorem 4.15. *Let $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ be a nano $b-I-$continuous function and $W \in \zeta_R(D)$. Then the restriction $f_{|W} : (W, \zeta_R(D)_{|W}, I_{|W}) \to (V, \zeta_{R'}(E))$ is nano $b - I -$ continuous.*

Proof. Let G be any nano open set in $(V, \zeta_{R'}(E))$. Since f is nano $b-I-$ *continuous*, we have $f^{-1}(G) \in NbIO(U, \zeta_R(D))$. Since $W \in \zeta_R(D)$, by Theorem 4.14, we have $W \cap f^{-1}(G) \in NbIO(W, \tau_R(D)_{|W}, I_{|W})$. On the other hand, $(f_{|W})^{-1}(G) = W \cap f^{-1}(G)$ and $(f_{|W})^{-1}(G) \in NbIO(W, \tau_R(D)_{|W}, I_{|W})$. This shows that $f_{|W} : NbIO(W, \zeta_R(D)_{|W}, I_{|W}) \to (V, \zeta_{R'}(E))$ is nano $b-I-$ *continuous*. □

Definition 4.16. A nano ideal topological space $(U, \zeta_R(D), I)$ is said to be *nano* $b-I-$ *normal* if for each pair of non-empty disjoint nano closed subsets A and B of U, there exist two nano b-I-open subsets G and W of U such that $A \subseteq G$, $B \subseteq W$ and $G \cap W = \phi$.

Theorem 4.17. *If $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ is nano b-I-continuous, nano closed injection and V is nano normal, then $(U, \zeta_R(D), I)$ is nano b-I-normal.*

Proof. Let A and B be two disjoint nano closed subsets of U. Since f is nano closed and injective, $f(A)$ and $f(B)$ are disjoint nano closed subsets of $(V, \zeta_{R'}(E))$. Since V is nano normal, there exist two nano open subsets G and W of V such that $f(A) \subseteq G$, $f(B) \subseteq W$ and $G \cap W = \phi$. Now $f^{-1}(G)$ and $f^{-1}(W)$ are nano b-I-open in U with $A \subseteq f^{-1}(G)$, $B \subseteq f^{-1}(W)$ and $f^{-1}(G) \cap f^{-1}(W) = \phi$. Thus $(U, \zeta_R(D), I)$ is nano b-I-normal. □

Definition 4.18. A nano ideal topological space $(U, \zeta_R(D), I)$ is said to be *nano* $b-I-$ *connected* if U can't be written as a union of two disjoint nano b-I-open subsets of D.

Theorem 4.19. *The nano b-I-continuous image of a nano b-I-connected space is nano connected.*

Proof. Let $f : (U, \zeta_R(D), I) \to (V, \zeta_{R'}(E))$ be a nano b-I-continuous function of a nano b-I-connected space $(U, \zeta_R(D), I)$ onto a nano topological space $(V, \zeta_{R'}(E))$. Assume that V is not nano connected, then $V = A \cup B$ where A and B are non-empty nano clopen with $A \cap B = \phi$. Since f is nano b-I-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty nano b-I-open in U. Also, $U = f^{-1}(V) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$. Hence U is not nano b-I-connected which is a contradiction. Therefore, V is nano connected. □

5. Nano b-I-open functions

Recall that a subset F of a nano ideal topological space $(U, \zeta_R(D), I)$ is said to be *nano* $semi-I-closed$ [26] if its complement is nano semi-I-open.

Definition 5.1. A function $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ is called *nano* $semi-I-open$ (resp.,*nano* $semi-I-closed$) if the image of every nano open (resp., nano closed) set in $(U, \zeta_R(D))$ is nano semi-I-open (resp., nano semi-I-closed) in $(V, \tau_{R'}(E), I)$.

Definition 5.2. A function $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ is called *nano* $b-I-open$ (resp., *nano* $b - I - closed$) if the image of every nano open (resp., nano closed) set in $(U, \zeta_R(D))$ is nano b-I-open (resp., nano b-I-closed) in $(V, \zeta_{R'}(E), I)$.

Remark. *Every nano semi-I-open (resp., nano semi-I-closed) function is nano b-I-open (resp., nano b-I-closed).*

The converse of the above remark need not be true as shown in the following example.

Example 5.3. Let $U = \{a, b, c, d\}$ be the universe, $D = \{a, b, d\} \subseteq U$, $U/R = \{\{a\}, \{b, d\}, \{c\}\}$, $\zeta_R(D) = \{\phi, U, \{a, b, d\}\}$ and let $V = \{a, b, c, d\}$, $E = \{b, d\} \subseteq V$, $V/R' = \{\{a, d\}, \{b\}, \{c\}\}$, $\zeta_{R'}(E)) = \{\phi, V, \{a, d\}, \{b\}, \{a, b, d\}\}$ and the ideal $I = \{\phi, \{a\}\}$. Define $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ by $f(a) = c$, $f(b) = b$, $f(c) = a$, $f(d) = d$. Then f is nano b-I-open but not nano semi I-open.

Theorem 5.4. *A function $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ is nano b-I-open if and only if for each $x \in U$ and each nano neighborhood W of x there exists $G \in NbIO(V, \zeta_{R'}(E))$ containing $f(x)$ such that $G \subseteq f(W)$.*

Proof. $\Rightarrow$) Suppose that f is a nano $b-I-open$ function. For each $x \in U$ and each nano neighborhood W of x, there exists $W_x \in \zeta_R(D)$ such that $x \in W_x \subseteq W$. Let $G = f(W_x)$. Since f is nano b-I-open, $G \in NbIO(V, \zeta_{R'}(E))$ and $f(x) \in G \subseteq f(W)$.

$\Leftarrow$) Let W be a nano open set in $(U, \zeta_R(D))$. For each $x \in W$, there exists $G_x \in NbIO(V, \zeta_{R'}(E))$ such that $f(x) \in G_x \subseteq f(W)$. Now $f(W) = \cup\{G_x : x \in W\}$ and so $f(W) \in NbIO(V, \zeta_{R'}(E))$. This shows that f is nano b-I-open. □

Theorem 5.5. *Let $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ be a nano b-I-open function. If G is any subset of V and C is a nano closed subset of U with $f^{-1}(G) \subseteq C$, then there exists a nano b-I-closed subset H of V with $G \subseteq H$ such that $f^{-1}(H) \subseteq C$.*

Proof. Suppose that f is a nano b-I-open function. Let G be any subset of V and C a nano closed subset of U with $f^{-1}(G) \subseteq C$. Then $U - C$ is nano open. Since f is nano b-I-open, $f(U - C)$ is nano b-I-open in V. Let $H = V - f(U - C)$. Then H is nano b-I-closed in V. Since $f^{-1}(G) \subseteq C$, $G \subseteq H$. Also, we obtain $f^{-1}(H) \subseteq C$. □

Theorem 5.6. *Let $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ be nano b-I-closed. If G is any subset of V and W is a nano open subset of U with $f^{-1}(G) \subseteq W$, then there exists a nano b-I-open subset H of V with $G \subseteq H$ such that $f^{-1}(H) \subseteq W$.*

Proof. Similar to that used in Theorem 5.6. □

Theorem 5.7. *For any bijective function $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$, the following are equivalent:*

1) *$f^{-1} : (V, \zeta_{R'}(E), I) \to (U, \zeta_R(D))$ is nano b-I-continuous.*
2) *f is nano b-I-open.*
3) *f is nano b-I-closed.*

Proof. It is straightforward. □

Theorem 5.8. *Let* $f : (U, \zeta_R(D)) \to (V, \zeta_{R'}(E), I)$ *and* $g : (V, \zeta_{R'}(E), I) \to (W, \zeta_{R"}(Z), K)$.

1) gof is nano b-I-open if f is nano open and g is a nano b-I-open.

2) f is nano b-I-open if gof is nano open and g is a nano b-I-continuous injection.

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Jamal M. Mustafa
email: `jjmmrr971@yahoo.com`
Department of Mathematics
Al al-Bayt University
Mafraq
JORDAN

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