Journal of
Mathematics
and Applications

JMA No 41, pp 29-38 (2018)

On a Cubic Integral Equation of Urysohn Type with Linear Perturbation of Second Kind

Hamed Kamal Awad, Mohamed Abdalla Darwish and Mohamed M.A. Metwali

Abstract: In this paper, we concern by a very general cubic integral equation and we prove that this equation has a solution in $C[0,1]$. We apply the measure of noncompactness introduced by Banaś and Olszowy and Darbo's fixed point theorem to establish the proof of our main result.

AMS Subject Classification: 45G10, 45M99, 47H09.
Keywords and Phrases: Cubic integral equation; Darbo's fixed point theorem; Monotonicity measure of noncompactness.

1. Introduction

Cubic integral equations have several useful applications in modeling numerous problems and events of the real world (cf. [3, 8, 9, 12, 13, 18, 19]).

In this paper we consider the cubic Urysohn integral equation with linear perturbation of second kind

$$x(\tau) = \phi(\tau) + \varphi(\tau, x(\tau)) + x^2(\tau) \int_0^1 u(\tau, s, (\Lambda x)(s))\, ds,\ \tau \in I = [0,1]. \tag{1.1}$$

In the above equation, we consider $\phi : I \to \mathbb{R}$, $\varphi : I \times \mathbb{R} \to \mathbb{R}$, $u : I \times I \times \mathbb{R} \to \mathbb{R}$ are given functions and $\Lambda : C(I) \to C(I)$ is an operator verifies special assumption which will state in Section 3.

Eq.(1.1) is of interest since it contains many includes several integral equations studied earlier as special cases, see [1, 2, 6, 7, 10, 11, 14, 15, 16, 20, 21, 22] and references therein. By using the measure of noncompactness related to monotonicity associated with fixed point theorem due to Darbo, we show that Eq.(1.1) has at least one solution in $C(I)$ which is nondecreasing on the interval I.

2. Auxiliary Facts and Results

In this section, we present some definitions and results which we will use further on.

Let $(E, \|\cdot\|)$ be a real Banach space with zero element 0. Let $B(x,r)$ be the closed ball centered at x with radius r. We denote by B_r the closed ball $B(0,r)$. Next, let X be a subset of E, we denote by $\overline{X}$ and Conv$X$ the closure and convex closure of X, respectively. We use the symbols λX and $X+Y$ for the usual algebraic operations on the sets. Moreover, the symbol $\mathfrak{M}_E$ stands for the family of all nonempty and bounded subsets of E and the symbol $\mathfrak{N}_E$ stands for its subfamily consisting of all relatively compact subsets.

Now, we state the definition of a measure of noncompactness [4]:

Definition 2.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is called a measure of noncompactness in E if it verifies the following assumptions:

(1) The family $\ker\mu \neq \emptyset$ and $\ker\mu \subset \mathfrak{N}_E$, where $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$.

(2) $\mu(X) \leq \mu(Y)$, if $X \subset Y$.

(3) $\mu(\overline{X}) = \mu(X)$ and $\mu(\text{Conv}X) = \mu(X)$.

(4) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y),\ 0 \leq \lambda \leq 1$.

(5) If $X_n \in \mathfrak{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ for $n = 1,\ 2,\ 3,\ \ldots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then $\cap_{n=1}^{\infty} X_n \neq \emptyset$.

Notice that $\ker\mu$ is said to be the kernel of the measure of noncompactness μ.

In the following, we will work in the Banach space $C(I)$ of all real functions defined and continuous on $I = [0,1]$ equipped with the standard norm $\|x\| = \max\{|x(\tau)| : \tau \in I\}$. We recall the measure of noncompactness in $C(I)$ which we will need in the next section (see [5]).

Let $\emptyset \neq X \subset C(I)$. For $x \in X$ and $\varepsilon \geq 0$ we denote by $\omega(x,\varepsilon)$ the modulus of continuity of the function x as follows

$$\omega(x,\varepsilon) = \sup\{|x(\tau) - x(t)| : \tau, t \in I,\ |\tau - t| \leq \varepsilon\}.$$

Next, we put $\omega(X,\varepsilon) = \sup\{\omega(x,\varepsilon) : x \in X\}$ and $\omega_0(X) = \lim_{\varepsilon\to 0} \omega(X,\varepsilon)$. Moreover, we define

$$d(x) = \sup\{|x(\tau) - x(t)| - [x(\tau) - x(t)] : \tau, t \in I,\ \tau \geq t\}$$

and

$$d(X) = \sup\{d(x) : x \in X\}.$$

Notice that $d(X) = 0$ if and only if all functions belonging to X are nondecreasing on I.

Finally, we define the function μ on the family $\mathfrak{M}_{C(I)}$ as follows

$$\mu(X) = \omega_0(X) + d(X).$$

Notice that the function μ is a measure of noncompactness in $C(I)$ [5].

We present a fixed point theorem due to Darbo [17] which we will need in the proof of our main result. First, we make use of the following definition.

Definition 2.2. Let $\emptyset \neq M$ be a subset of a Banach space E and let $\mathfrak{P} : M \to E$ be a continuous mapping which maps bounded sets onto bounded sets. The operator $\mathfrak{P}$ satisfies the Darbo condition (with a constant $\kappa \geq 0$) with respect to a measure of noncompactness μ if for any bounded subset X of M we have

$$\mu(\mathfrak{P}X) \leq \kappa\mu(X).$$

If $\mathfrak{P}$ verifies the Darbo condition with $\kappa < 1$ then it is a contraction operator with respect to μ.

Theorem 2.3. *Let $\emptyset \neq \Omega$ be a closed, bounded and convex subset of the space E and let $\mathfrak{P} : \Omega \to \Omega$ be a contraction mapping with respect to the measure of noncompactness μ.*
Then $\mathfrak{P}$ has a fixed point in the set Ω.

Notice that the assumptions of the above theorem gives us that the set $\text{Fix}\mathfrak{P}$ of all fixed points of $\mathfrak{P}$ belongs to Ω is an element of $\ker\mu$ [4].

3. The Main Result

We consider Eq.(1.1) and assume that the following assumptions are verified:

(a_1) The function $\phi : I \to \mathbb{R}$ is continuous, nonnegative and nondecreasing on I.

(a_2) The function $\varphi : I \times \mathbb{R} \to \mathbb{R}$ is continuous, $\varphi : I \times \mathbb{R}_+ \to \mathbb{R}_+$ and

$$\exists\, c \geq 0 : \ |\varphi(\tau, x_1) - \varphi(\tau, x_2)| \leq c|x_1 - x_2| \quad \forall\ (x_1, x_2) \in \mathbb{R}^2\ \&\ \tau \in I.$$

(a_3) The superposition operator Φ generated by the function φ satisfies for any nonnegative function x the condition $d(\Phi x) \leq cd(x)$, where c is the same c appears in assumption (a_2).

(a_4) The function $u : I \times I \times \mathbb{R} \to \mathbb{R}$ is a continuous, $u : I \times I \times \mathbb{R}_+ \to \mathbb{R}_+$ and for arbitrary fixed $t \in I$ and $x \in \mathbb{R}$ the function $\tau \to u(\tau, t, x)$ is nondecreasing on I. Moreover,

$$\exists\ \Psi : \mathbb{R}_+ \to \mathbb{R}_+(\text{nondecreasing}) : \ |u(\tau, t, x)| \leq \Psi(|x|) \quad \forall\ (\tau, t) \in I^2\ \&\ x \in \mathbb{R}.$$

(a_5) The operator $\Lambda : C(I) \to C(I)$ is continuous and

$$\exists\ \psi : \mathbb{R}_+ \to \mathbb{R}_+(\text{nondecreasing}) : \ |(\Lambda x)(\tau)| \leq \psi(\|x\|) \text{ for any } \tau \in I,\ x \in C(I).$$

Moreover, for every nonnegative function $x \in C(I)$, the function Λx is nonnegative and nondecreasing on I.

(a_6) The inequality

$$\|\phi\| + cr + \varphi^* + r^2\Psi(\psi(r)) \leq r \tag{3.1}$$

has a positive solution r_0 such that $c+2r_0\Psi(\psi(r_0)) < 1$, where $\varphi^* = \max_{0\leq\tau\leq 1} \varphi(\tau, 0)$.

Under the above assumptions, we state our main result as follows.

Theorem 3.1. *Let the assumptions $(a_1) - (a_6)$ be verified, then the cubic Urysohn integral equation (1.1) has at least one solution $x \in C(I)$ which is nondecreasing on I.*

Proof. Let $\mathfrak{F}$ be an operator defined on $C(I)$ by

$$(\mathfrak{F}x)(\tau) = \phi(\tau) + \varphi(\tau, x(\tau)) + x^2(\tau)(\mathcal{U}x)(t), \tag{3.2}$$

where $\mathcal{U}$ is the Urysohn integral operator

$$(\mathcal{U}x)(\tau) = \int_0^1 u(\tau, t, (\Lambda x)(t))\, dt. \tag{3.3}$$

For better readability, we will write the proof in seven steps.

Step 1: $\mathfrak{F}$ *maps the space $C(I)$ into itself.*

Notice that for a given $x \in C(I)$, according to assumptions $(a_1) - (a_5)$, we have $\mathfrak{F}x \in C(I)$. Therefore, the operator $\mathfrak{F}$ maps $C(I)$ into itself.

Step 2: $\mathfrak{F}$ *maps the ball B_{r_0} into itself.*

For all $\tau \in I$, we have

$$\begin{aligned} |(\mathfrak{F}x)(\tau)| &\leq \left|\phi(\tau) + \varphi(\tau, x(\tau)) + x^2(\tau)\int_0^1 u(\tau, t, (\Lambda x)(t))\, dt\right| \\ &\leq |\phi(\tau)| + |\varphi(\tau, x(\tau)) - \varphi(\tau, 0)| + |\varphi(\tau, 0)| \\ &\quad + |x^2(\tau)| \int_0^1 |u(\tau, t, (\Lambda x)(t))|\, dt \\ &\leq \|\phi\| + c\|x\| + \varphi^* + \|x\|^2\Psi(\psi(\|x\|)) \int_0^1 ds \\ &= \|\phi\| + c\|x\| + \varphi^* + \|x\|^2\Psi(\psi(\|x\|)). \end{aligned}$$

From the above estimate, we get

$$\|\mathfrak{F}x\| \leq \|\phi\| + c\|x\| + \varphi^* + \|x\|^2\Psi(\psi(\|x\|)).$$

Therefore, if we have $\|x\| \leq r_0$, we obtain

$$\|\mathfrak{F}x\| \leq \|\phi\| + cr_0 + \varphi^* + r_0^2\Psi(\psi(r_0)) \leq r_0,$$

in view of the assumption (a_6). Consequently, the operator $\mathfrak{F}$ maps the ball B_{r_0} into itself.

Further, let $B^+_{r_0}$ be the subset of B_{r_0} given by

$$B^+_{r_0} = \{x \in B_{r_0} : x(\tau) \geq 0, \text{ for } \tau \in I\}.$$

Notice that, the set $\emptyset \neq B^+_{r_0}$ is closed, bounded and convex.

Step 3: *$\mathfrak{F}$ maps continuously the ball $B^+_{r_0}$ into itself.*

In view of the above facts about $B^+_{r_0}$ and assumptions $(a_1) - (a_4)$, we infer that $\mathfrak{F}$ maps the set $B^+_{r_0}$ into itself.

Step 4: *The operator $\mathfrak{F}$ is continuous on $B^+_{r_0}$.*

To establish this, let us fix arbitrarily $\varepsilon > 0$ and $y \in B^+_{r_0}$. By assumption (a_4), we can find $\delta > 0$ such that for arbitrary $x \in B^+_{r_0}$ with $\|x - y\| \leq \delta$ we have that $\|\Lambda x - \Lambda y\| \leq \varepsilon$. Indeed, for each $\tau \in I$ we have

$$\begin{aligned}
&|(\mathfrak{F}x)(\tau) - (\mathfrak{F}y)(\tau)| \\
&\leq |\varphi(\tau, x(\tau)) - \varphi(\tau, y(\tau))| \\
&\quad + \left| x^2(\tau) \int_0^1 u(\tau, t, (\Lambda x)(t))\, dt - y^2(\tau) \int_0^1 u(\tau, t, (\Lambda y)(t))\, dt \right| \\
&\leq c|x(\tau) - y(\tau)| + \left| x^2(\tau) \int_0^1 u(\tau, t, (\Lambda x)(t))\, dt - y^2(\tau) \int_0^1 u(\tau, t, (\Lambda x)(t))\, dt \right| \\
&\quad + \left| y^2(\tau) \int_0^1 u(\tau, t, (\Lambda x)(t))\, dt - y^2(\tau) \int_0^1 u(\tau, t, (\Lambda y)(t))\, dt \right| \\
&\leq c|x(\tau) - y(\tau)| + |x^2(\tau) - y^2(\tau)| \int_0^1 |u(\tau, t, (\Lambda x)(t))|\, dt \\
&\quad + |y^2(\tau)| \int_0^1 |u(\tau, t, (\Lambda x)(t)) - u(\tau, t, (\Lambda y)(t))|\, dt.
\end{aligned}$$

Therefore, we have

$$\|\mathfrak{F}x - \mathfrak{F}y\| \leq c\|x - y\| + 2r_0 \Psi(\psi(r_0))\|x - y\| + r_0^2 \omega^*(u, \varepsilon), \tag{3.4}$$

where we denoted

$$\omega^*(u, \varepsilon) = \sup\{|u(\tau, t, x) - u(\tau, t, y)| : \tau, t \in I,\ x, y \in [0, \psi(r_0)],\ |x - y| \leq \varepsilon\}.$$

From assumption (a_4) we infer that $\omega^*(u, \varepsilon) \to 0$ as $\varepsilon \to 0$ and therefore, the operator $\mathfrak{F}$ is continuous in $B^+_{r_0}$.

Step 5: *An estimate of $\mathfrak{F}$ with respect to the term related to continuity ω_0.*

Let $\emptyset \neq X \subset B^+_{r_0}$, fix an arbitrarily number $\varepsilon > 0$ and choose $x \in X$ and $\tau_1, \tau_2 \in I$ such that $|\tau_2 - \tau_1| \leq \varepsilon$. Without restriction of the generality, we may assume that $\tau_1 \leq \tau_2$. In the view of our assumptions, we have

$$
\begin{aligned}
&|(\mathfrak{F}x)(\tau_2) - (\mathfrak{F}x)(\tau_1)| \\
&\le |\phi(\tau_2) - \phi(\tau_1)| + |\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))| \\
&\quad + \left|x^2(\tau_2)\,(\mathcal{U}x)(\tau_2) - x^2(\tau_2)\,(\mathcal{U}x)(\tau_1)\right| \\
&\quad + \left|x^2(\tau_2)\,(\mathcal{U}x)(\tau_1) - x^2(\tau_1)\,(\mathcal{U}x)(\tau_1)\right| \\
&\le \omega(\phi,\varepsilon) + |\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_2))| + |\varphi(\tau_1, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))| \\
&\quad + \left|x^2(\tau_2)\right|\ |(\mathcal{U}x)(\tau_2) - (\mathcal{U}x)(\tau_1)| + \left|x^2(\tau_2) - x^2(\tau_1)\right|\ |(\mathcal{U}x)(\tau_1)| \\
&\le \omega(\phi,\varepsilon) + \gamma_{r_0}(\varphi,\varepsilon) + c\,\omega(x,\varepsilon) + |x(\tau_2)|^2\ |(\mathcal{U}x)(\tau_2) - (\mathcal{U}x)(\tau_1)| \\
&\quad + |x(\tau_2) - x(\tau_1)|\ |x(\tau_2) + x(\tau_1)|\ |(\mathcal{U}x)(\tau_1)| \\
&\le \omega(\phi,\varepsilon) + \gamma_{r_0}(\varphi,\varepsilon) + c\,\omega(x,\varepsilon) \\
&\quad + \|x\|^2 \int_0^1 |u(\tau_2, t, (\Lambda x)(t)) - u(\tau_1, t, (\Lambda x)(t))|\,dt + 2\|x\|\omega(x,\varepsilon)\Psi(\psi(\|x\|)) \\
&\le \omega(\phi,\varepsilon) + \gamma_{r_0}(\varphi,\varepsilon) + c\,\omega(x,\varepsilon) + \|x\|^2\omega_{\psi(\|x\|)}(u,\varepsilon) + 2\|x\|\omega(x,\varepsilon)\Psi(\psi(\|x\|)),
\end{aligned}
$$

where we denoted

$$\gamma_{r_0}(\varphi,\varepsilon) = \sup\left\{|\varphi(\tau_2, x) - \varphi(\tau_1, x)| : \tau_1, \tau_2 \in I,\ x \in [0, r_0],\ |\tau_2 - \tau_1| \le \varepsilon\right\}$$

and

$$\omega_b(u,\varepsilon) = \sup\left\{|u(\tau_2, t, y) - u(\tau_1, t, y)| : t, \tau_1, \tau_2 \in I,\ y \in [0, b],\ |\tau_2 - \tau_1| \le \varepsilon\right\}.$$

Hence,

$$\omega(\mathfrak{F}x,\varepsilon) \le \omega(\phi,\varepsilon) + \gamma_{r_0}(\varphi,\varepsilon) + c\,\omega(x,\varepsilon) + r_0^2\omega_{\psi(r_0)}(u,\varepsilon) + 2r_0\omega(x,\varepsilon)\Psi(\psi(r_0)).$$

Consequently,

$$\omega(\mathfrak{F}X,\varepsilon) \le \omega(\phi,\varepsilon) + \gamma_{r_0}(\varphi,\varepsilon) + (c + 2r_0\Psi(\psi(r_0)))\,\omega(X,\varepsilon) + r_0^2\omega_{\psi(r_0)}(u,\varepsilon).$$

Since the function ϕ is continuous on I, the function φ is uniformly continuous on $I \times [0, r_0]$ and the function u is uniformly continuous the set $I \times I \times [0, \psi(r_0)]$, then we obtain

$$\omega_0(\mathfrak{F}X) \le (c + 2r_0\Psi(\psi(r_0)))\,\omega_0(X). \tag{3.5}$$

Step 6: *An estimate of $\mathfrak{F}$ with respect to the term related to monotonicity d.*

Fix an arbitrary $x \in X$ and $\tau_1, \tau_2 \in I$ with $\tau_2 > \tau_1$. Then, taking into account our assumption, we get

$$
\begin{aligned}
&|(\mathfrak{F}x)(\tau_2) - (\mathfrak{F}x)(\tau_1)| - ((\mathfrak{F}x)(\tau_2) - (\mathfrak{F}x)(\tau_1)) \\
&= \left|\phi(\tau_2) + \varphi(\tau_2, x(\tau_2)) + x^2(\tau_2) \int_0^1 u(\tau_2, t, (\Lambda x)(t))\,dt\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\phi(\tau_1)-\varphi(\tau_1,x(\tau_1))-x^2(\tau_1)\int_0^1 u(\tau_1,t,(\Lambda x)(t))\,dt\bigg|\\
&-\bigg(\phi(\tau_2)+\varphi(\tau_2,x(\tau_2))+x^2(\tau_2)\int_0^1 u(\tau_2,t,(\Lambda x)(t))\,dt\\
&-\phi(\tau_1)-\varphi(\tau_1,x(\tau_1))-x^2(\tau_1)\int_0^1 u(\tau_1,t,(\Lambda x)(t))\,dt\bigg)\\
\le{}& [|\phi(\tau_2)-\phi(\tau_1)|-(\phi(\tau_2)-\phi(\tau_1))]\\
&+[|\varphi(\tau_2,x(\tau_2))-\varphi(\tau_1,x(\tau_1))|-(\varphi(\tau_2,x(\tau_2))-\varphi(\tau_1,x(\tau_1)))]\\
&+\left|x^2(\tau_2)\int_0^1 u(\tau_2,t,(\Lambda x)(t))\,dt-x^2(\tau_1)\int_0^1 u(\tau_2,t,(\Lambda x)(t))\,dt\right|\\
&+\left|x^2(\tau_1)\int_0^1 u(\tau_2,t,(\Lambda x)(t))\,dt-x^2(\tau_1)\int_0^1 u(\tau_1,t,(\Lambda x)(t))\,dt\right|\\
&-\left(x^2(\tau_2)\int_0^1 u(\tau_2,t,(\Lambda x)(t))\,dt-x^2(\tau_1)\int_0^1 u(\tau_2,t,(\Lambda x)(t))\,dt\right)\\
&-\left(x^2(\tau_1)\int_0^1 u(\tau_2,t,(\Lambda x)(t))\,dt-x^2(\tau_1)\int_0^1 u(\tau_1,t,(\Lambda x)(t))\,dt\right)\\
\le{}& |\varphi(\tau_2,x(\tau_2))-\varphi(\tau_1,x(\tau_1))|-(\varphi(\tau_2,x(\tau_2))-\varphi(\tau_1,x(\tau_1)))\\
&+\left[|x^2(\tau_2)-x^2(\tau_1)|-(x^2(\tau_2)-x^2(\tau_1))\right]\int_0^1 u(\tau_2,t,(\Lambda x)(t))\,dt\\
&+x^2(\tau_1)\left[\left|\int_0^1 u(\tau_2,t,(\Lambda x)(t))\,dt-\int_0^1 u(\tau_1,t,(\Lambda x)(t))\,dt\right|\right.\\
&\left.-\left(\int_0^1 u(\tau_2,t,(\Lambda x)(t))\,dt-\int_0^1 u(\tau_1,t,(\Lambda x)(t))\,dt\right)\right]\\
\le{}& d(\Phi x)+2\|x\|\Psi(\psi(\|x\|))d(x).
\end{aligned}
$$

The above estimate gives us that

$$d(\mathfrak{F}x)\le cd(x)+2r_0\Psi(\psi(r_0))d(x),$$

and consequently,

$$d(\mathfrak{F}X)\le (c+2r_0\Psi(\psi(r_0)))d(X). \tag{3.6}$$

Step 7: $\mathfrak{F}$ *is a contraction with respect to the measure of noncompactness* μ.

By adding (3.5) and (3.6), we get

$$\omega_0(\mathfrak{F}X)+d(\mathfrak{F}X)\le (c+2r_0\Psi(\psi(r_0)))\omega_0(X)+(c+2r_0\Psi(\psi(r_0)))d(X)$$

or

$$\mu(\mathfrak{F}X)\quad\le\quad (c+2r_0\Psi(\psi(r_0)))\mu(X).$$

Since $c + 2r_0\Psi(\psi(r_0)) < 1$, then the operator $\mathfrak{F}$ is contraction with respect to the measure of noncompactness μ.

Finally, Theorem 2.3 guarantees that Eq.(1.1) has at least one solution $x \in C(I)$ which is nondecreasing on I. This completes the proof. $\square$

4. Example

Let us consider the cubic Urysohn integral equation

$$x(\tau) = \frac{\sqrt{\tau}}{8} + \frac{\tau x(\tau)}{1+\tau^2} + \frac{x^2(\tau)}{4}\int_0^1 \arctan\left(\frac{\tau\int_0^t s x^2(s)\,ds}{1+t^2}\right)\,dt. \tag{4.1}$$

Here, $\phi(\tau) = \frac{\sqrt{\tau}}{8}$ and this function verifies assumption (a_1) and $\|\phi\| = 1/8$. Also, $\varphi(\tau, x) = \frac{\tau x}{1+\tau^2}$ and this function verifies assumption (a_2) with

$$|\varphi(\tau, x) - \varphi(\tau, y)| \leq \frac{1}{2}|x - y| \quad \forall\, t \in I \;\&\; (x, y) \in \mathbb{R}^2.$$

Moreover, the function φ verifies assumption (a_3). Indeed, for arbitrary nonnegative function $x \in C(I)$ and $\tau_1, \tau_2 \in I$ with $\tau_1 \leq \tau_2$, we have

$$\begin{aligned}
d(\Phi x) &= |(\Phi x)(\tau_2) - (\Phi x)(\tau_1)| - ((\Phi x)(\tau_2) - (\Phi x)(\tau_1)) \\
&= |\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))| - (\varphi(\tau_2, x(\tau_2)) - \varphi(\tau_1, x(\tau_1))) \\
&= \left|\frac{\tau_2}{1+\tau_2^2}x(\tau_2) - \frac{\tau_1}{1+\tau_1^2}x(\tau_1)\right| - \left(\frac{\tau_2}{1+\tau_2^2}x(\tau_2) - \frac{\tau_1}{1+\tau_1^2}x(\tau_1)\right) \\
&\leq \frac{\tau_2}{1+\tau_2^2}\,|x(\tau_2) - x(\tau_1)| + \left|\frac{\tau_2}{1+\tau_2^2} - \frac{\tau_1}{1+\tau_1^2}\right| x(\tau_1) \\
&\quad - \frac{\tau_2}{1+\tau_2^2}\,(x(\tau_2) - x(\tau_1)) - \left(\frac{\tau_2}{1+\tau_2^2} - \frac{\tau_1}{1+\tau_1^2}\right) x(\tau_1) \\
&= \frac{\tau_2}{1+\tau_2^2}\,[|x(\tau_2) - x(\tau_1)| - (x(\tau_2) - x(\tau_1))] \\
&= \frac{\tau_2}{1+\tau_2^2} d(x) \;\leq\; \frac{1}{2} d(x).
\end{aligned}$$

The function $u(\tau, t, x) = \arctan \frac{\tau x}{1+t^2}$ satisfies assumption (a_4), we have $|u(\tau, t, x)| \leq |x|$ which means $\Psi(r) = r$. Moreover, the operator $(\Lambda x)(\tau) = \int_0^\tau t x^2(t)\,dt$ verifies assumption (a_5) with $\psi(r) = r^2$.

Therefore, the inequality (3.1) has the form $\frac{1}{8} + \frac{r}{2} + r^4 \leq r$ or $\frac{1}{4} + r + 2r^4 \leq 2r$. This inequality admits $r_0 = 1/2$ as a positive solution. Moreover,

$$c + 2r_0\Psi(\psi(r_0)) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < 1.$$

Consequently, Theorem 3.1 guarantees that equation (4.1) has a continuous nondecreasing solution.

References

[1] J. Appell, *Implicit functions, nonlinear integral equations, and the measure of noncompactness of the superposition operator*, J. Math. Anal. Appl. 3 (1981) 251–263.

[2] J. Appell, C. Chen, *How to solve Hammerstein equations*, J. Integral Equations Appl. 18 (2006) 287–296.

[3] H.K. Awad, M.A. Darwish, *On monotonic solutions of a cubic Urysohn Integral equation with linear modification of the argument*, Adv. Dyn. Syst. Appl. 13 (2018) 91–99.

[4] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics 60 Marcel Dekker, New York, 1980.

[5] J. Banaś, L. Olszowy, *Measures of noncompactness related to monotonicity*, Comment. Math. 41 (2001) 13–23.

[6] M. Benchohra, M.A. Darwish, *On unique solvability of quadratic integral equations with linear modification of the argument*, Miskolc Math. Notes 10 (2009) 3–10.

[7] T.A. Burton, *Volterra Integral and Differential Equations*, Academic Press, New York, 1983.

[8] K.M. Case, P.F. Zweifel, *Linear Transport Theory*, Addison-Wesley, Reading, MA, 1967.

[9] J. Caballero, D. O'Regan, K. Sadarangani, *On nondecreasing solutions of cubic integral equations of Urysohn type*, Comment. Math. (Prace Mat.) 44 (2004) 39–53.

[10] M.A. Darwish, *On integral equations of Urysohn-Volterra type*, Appl. Math. Comput. 136 (2003) 93–98.

[11] M.A. Darwish, *On quadratic integral equation of fractional orders*, J. Math. Anal. Appl. 311 (2005) 112–119.

[12] M.A. Darwish, *On monotonic solutions of a singular quadratic integral equation with supremum*, Dynam. Syst. Appl. 17 (2008) 539–549.

[13] K. Deimling, *Nonlinear Fuctional Analysis*, Springer-Verlag, Berlin, 1985.

[14] W.G. El-Sayed, A.A. El-Bary, M.A. Darwish, *Solvability of Urysohn integral equation*, Appl. Math. Comput. 145 (2003) 487–493.

[15] W.G. El-Sayed, B. Rzepka, *Nondecreasing solutions of a quadratic integral equation of Urysohn type*, Comput. Math. Appl. 51 (2006) 1065–1074.

[16] D. Franco, G. Infante, D. O'Regan, *Positive and nontrivial solutions for the Urysohn integral equation*, Acta Math. Sin. (Engl. Ser.) 22 (2006) 1745–1750.

[17] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[18] S. Hu, M. Khavani, W. Zhuang, *Integral equations arrising in the kinetic theory of gases*, Appl. Anal. 34 (1989) 261–266.

[19] C.T. Kelly, *Approximation of solutions of some quadratic integral equations in transport theory*, J. Integral Eq. 4 (1982) 221–237.

[20] D. O'Regan, M. Meehan, *Existence Theory for Nonlinear Integral and Integrodifferential Equations*, Kluwer Academic Publishers, Dordrecht, 1998.

[21] M. Väth, *Volterra and Integral Equations of Vector Functions*, Monographs and Textbooks in Pure and Applied Mathematics 224, Marcel Dekker, Inc., New York, 2000.

[22] P.P. Zabrejko et al., *Integral Equations - a Reference Text*, Noordhoff International Publishing, The Netherlands 1975 (Russian edition: Nauka, Moscow, 1968).

DOI: 10.7862/rf.2018.3

Hamed Kamal Awad
email: hamedk66@sci.dmu.edu.eg
ORCID: 0000-0003-4866-3164
Department of Mathematics, Faculty of Science
Damanhour University
Damanhour
EGYPT

Mohamed Abdalla Darwish
email: dr.madarwish@gmail.com
ORCID: 0000-0002-4245-4364
Department of Mathematics, Faculty of Science
Damanhour University
Damanhour
EGYPT

Mohamed M.A. Metwali
email: m.metwali@yahoo.com
ORCID: 0000-0003-1091-8619
Department of Mathematics, Faculty of Science
Damanhour University
Damanhour
EGYPT

Received 09.11.2018 *Accepted 11.12.2018*