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An Upper Bound for Third Hankel Determinant of Starlike Functions Related to Shell-like Curves Connected with Fibonacci Numbers

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Abstract: We investigate the third Hankel determinant problem for some starlike functions in the open unit disc, that are related to shell-like curves and connected with Fibonacci numbers. For this, firstly, we prove a conjecture, posed in [17], for sharp upper bound of second Hankel determinant. In the sequel, we obtain another sharp coefficient bound which we apply in solving the problem of the third Hankel determinant for these functions.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are *analytic* in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \,\text{and}\, |z| < 1\}$ and let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and are of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We say that f is subordinate to F in $\mathbb{U}$, written as $f \prec F$, if and only if $f(z) = F(w(z))$ for some analytic function w such that $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$.

If $f \in \mathcal{A}$ and

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad \text{or} \quad 1 + \frac{zf''(z)}{f'(z)} \prec p(z)$$

where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike or convex respectively. These functions form known classes denoted by $\mathcal{S}^*$ or $\mathcal{C}$ respectively. These classes are very important subclasses of the class $\mathcal{S}$ in geometric function theory. In this paper we consider the following subclass of starlike functions.

Definition 1. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z)$$

with

$$\tilde{p}(z) = \frac{1 + \tau^2 z^2}{1 - \tau z - \tau^2 z^2},$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$. The class $\mathcal{SL}$ was introduced in [16].

The function $\tilde{p}$ is not univalent in $\mathbb{U}$, but it is univalent in the disc $|z| < \tau^2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1 - |\tau|},$$

which shows that the number $|\tau|$ divides $[0, 1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by

$$(10x - \sqrt{5})y^2 = (\sqrt{5} - 2x)(\sqrt{5}x - 1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \leq r_0 = (3 - \sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:

$$\tau^n = u_n \tau + u_{n-1}. \tag{1.2}$$

In 1976, Noonan and Thomas [10] stated the s^{th} Hankel determinant for $s \geq 1$ and $k \geq 1$ as

$$H_s(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+s-1} \\ a_{k+1} & a_{k+2} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+s-1} & \cdots & \cdots & a_{k+2(s-1)} \end{vmatrix} \tag{1.3}$$

where $a_1 = 1$.

This determinant has also been considered by several authors. For example, Noor [11] determined the rate of growth of $H_s(k)$ as $k \to \infty$ for functions f given by (1) with bounded boundary. Ehrenborg in [3] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [8]. Also, several authors considered the case $s = 2$. Especially, $H_2(1) = a_3 - a_2^2$ is known as Fekete-Szegö functional and this functional is generalized to $a_3 - \mu a_2^2$ where μ is some real number [4]. Estimating for an upper bound of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegö problem. In [13], Raina and Sokół considered Fekete-Szegö problem for the class $\mathcal{SL}$. In 1969, Keogh and Merkes [7] solved this problem for the classes $\mathcal{S}^*$ and $\mathcal{C}$. The second Hankel determinant is $H_2(2) = a_2a_4 - a_3^2$. Janteng [5] found the sharp upper bound for $|H_2(2)|$ for univalent functions whose derivative has positive real part. Also, in [6] Janteng et al. obtained the bounds for $|H_2(2)|$ for the classes $\mathcal{S}^*$ and $\mathcal{C}$. In [17], Sokół et al. considered second Hankel determinant problem for the class $\mathcal{SL}$ and obtained sharp upper bounds for the functional $|a_2a_4 - a_3^2|$ belonging to the class $\mathcal{SL}$. Also they gave a conjecture for sharp bound of $|a_2a_4 - a_3^2|$ for functions in the class $\mathcal{SL}$. The third Hankel determinant is $H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$. Recently, Babaloa [1], Raza and Malik [15] and Bansal et al. [2] have studied third Hankel determinant $H_3(1)$, for various classes of analytic and univalent functions.

In this paper, we investigate an upper bound on the modulus of $H_3(1)$ the functions belonging to the class $\mathcal{SL}$ of analytic functions related to shell-like curves connected with Fibonacci numbers in the open unit disc defined by (1.1).

Now we recall the following lemmas which will be use in proving our main results. Let $\mathcal{P}(\beta)$, $0 \le \beta < 1$, denote the class of analytic functions p in $\mathbb{U}$ with $p(0) = 1$ and $Re\{p(z)\} > \beta$. Especially, we will use $\mathcal{P}$ instead of $\mathcal{P}(0)$.

Lemma 1.1. *([12]) Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1z + c_2z^2 + \cdots$, then*

$$|c_n| \le 2, \qquad for \qquad n \ge 1. \tag{1.4}$$

If $|c_1| = 2$, then $p(z) \equiv p_1(z) \equiv (1 + xz)/(1 - xz)$ with $x = \frac{c_1}{2}$. Conversely, if $p(z) \equiv p_1(z)$ for some $|x| = 1$, then $c_1 = 2x$. Furthermore, we have

$$\left| c_2 - \frac{c_1^2}{2} \right| \le 2 - \frac{|c_1|^2}{2}. \tag{1.5}$$

If $|c_1| < 2$, and $\left| c_2 - \frac{c_1^2}{2} \right| = 2 - \frac{|c_1|^2}{2}$, then $p(z) \equiv p_2(z)$, where

$$p_2(z) = \frac{1 + \bar{x}wz + z(wz + x)}{1 + \bar{x}wz - z(wz + x)}$$

and $x = \frac{c_1}{2}$, $w = \frac{2c_2 - c_1^2}{4 - |c_1|^2}$ and $|c_2 - \frac{c_1^2}{2}| = 2 - \frac{|c_1|^2}{2}$.

Lemma 1.2. *([14]) Let $p \in \mathcal{P}$ with coefficients c_n as above, then*

$$|c_1c_2 - c_3| \le 2. \tag{1.6}$$

Lemma 1.3. *([9]) Let $p \in \mathcal{P}$ with coefficients c_n as above, then*

$$|c_3 - 2c_1c_2 + c_1^3| \leq 2. \tag{1.7}$$

Lemma 1.4. *([16]) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to the class $\mathcal{SL}$, then*

$$|a_n| \leq |\tau|^{n-1} u_n, \tag{1.8}$$

where u_n is the sequence of Fibonacci numbers and $\tau = \frac{1-\sqrt{5}}{2}$. Equality holds in (1.8) for the function $f_0(z) = \frac{z}{1-\alpha z-\alpha^2 z^2}$.

Lemma 1.5. *([13]) If $f(z) = z + \sum_{n=2} \infty a_n z^n$ belongs to the class $\mathcal{SL}$, then*

$$|a_3 - \lambda a_2^2| \leq \tau^2(2+\lambda) \text{ for all } \lambda \in \mathbb{C}. \tag{1.9}$$

The above estimation is sharp. If $\lambda > 0$, then the equality in (1.9) is attained by the function $f_0(z) = \frac{z}{1-\alpha z-\alpha^2 z^2}$ while by the function $-f_0(-z)$, when $\lambda \leq 0$.

Especially, when $\lambda = 1$ in (1.9), we obtain $|a_3 - a_2^2| \leq 3\tau^2$.

In this study, we use ideas and techniques used in geometric function theory. The central problem considered here is the sharp upper bounds for the functionals $|H_2(2)|$ and $|a_2a_3 - a_4|$ of functions in the class $\mathcal{SL}$ depicted by the Fibonacci numbers, respectively. Also the third Hankel determinant $|H_3(1)|$ is considered using these functionals.

2. Main Results

In [17] it was proved that if $f(z) = z + a_2z^2 + \ldots$ belongs to $\mathcal{SL}$, then

$$|H_2(2)| = |a_2a_4 - a_3^2| \leq \frac{11}{3}\tau^4.$$

And it was conjectured that $|H_2(2)| = |a_2a_4 - a_3^2| \leq \tau^4$. Firstly, we present a proof of this.

Theorem 2.1. *If $f(z) = z + a_2z^2 + \ldots$ belongs to $\mathcal{SL}$, then*

$$|H_2(2)| = |a_2a_4 - a_3^2| \leq \tau^4. \tag{2.1}$$

The bound is sharp.

Proof. For given $f \in \mathcal{SL}$, define $p(z) = 1 + p_1z + p_2z^2 + \cdots$, by

$$\frac{zf'(z)}{f(z)} = p(z) = 1 + p_1z + p_2z^2 + \cdots,$$

where $p \prec \tilde{p}$. If $p \prec \tilde{p}$, then there exists an analytic function w such that $|w(z)| \leq |z|$ in $\mathbb{U}$ and $p(z) = \tilde{p}(w(z))$. Therefore, the function

$$h(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots \tag{2.2}$$

is in the class $\mathcal{P}$. It follows that

$$w(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \tag{2.3}$$

and

$$\begin{aligned}
\tilde{p}(w(z)) = 1 &+ \tilde{p}_1 \left\{ \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \right\} \\
&+ \tilde{p}_2 \left\{ \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \right\}^2 \\
&+ \tilde{p}_3 \left\{ \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \right\}^3 + \cdots \\
= 1 &+ \frac{\tilde{p}_1 c_1 z}{2} + \left\{ \left(c_2 - \frac{c_1^2}{2}\right)\frac{\tilde{p}_1}{2} + \frac{c_1^2}{4}\tilde{p}_2 \right\} z^2 \\
&+ \left\{ \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{\tilde{p}_1}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{c_1 \tilde{p}_2}{2} + \frac{c_1^3}{8}\tilde{p}_3 \right\} z^3 + \cdots .
\end{aligned} \tag{2.4}$$

It is known that

$$\begin{aligned}
\tilde{p}(z) &= 1 + \sum_{n=1}^{\infty} \tilde{p}_n z^n \\
&= 1 + (u_0 + u_2)\tau z + (u_1 + u_3)\tau^2 z^2 + \sum_{n=3}^{\infty} (u_{n-3} + u_{n-2} + u_{n-1} + u_n)\tau^n z^n \\
&= 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + \cdots .
\end{aligned} \tag{2.5}$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_n + u_{n+1}$ for $n = 0, 1, 2, \ldots$. Thus, $\tilde{p}_1 = \tau$, $\tilde{p}_2 = 3\tau^2$ and

$$\tilde{p}_n = (u_{n-1} + u_{n+1})\tau^n = (u_{n-3} + u_{n-2} + u_{n-1} + u_n)\tau^n = \tau\tilde{p}_{n-1} + \tau^2\tilde{p}_{n-2} \quad (n = 3, 4, 5, \ldots).$$

If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, then using (2.4) and (2.5) , we have

$$p_1 = \frac{c_1}{2}\tau, \tag{2.6}$$

$$p_2 = \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)\tau + \frac{3}{4}c_1^2\tau^2, \tag{2.7}$$

and

$$p_3 = \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\tau + \frac{3}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tau^2 + \frac{1}{2}c_1^3\tau^3. \tag{2.8}$$

Hence

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + \cdots = 1 + p_1z + p_2z^2 + \cdots$$

and

$$a_2 = p_1, \quad a_3 = \frac{p_1^2 + p_2}{2}, \quad a_4 = \frac{p_1^3 + 3p_1p_2 + 2p_3}{6}.$$

Therefore, we have

$$\begin{aligned}
&|a_2a_4 - a_3^2|\\
=&\frac{1}{12}\left|p_1^4 - 4p_1p_3 + 3p_2^2\right|\\
=&\frac{1}{12}\left|\frac{c_1^4}{16}\tau^4 - 2c_1\tau\left\{\frac{\tau}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{3c_1\tau^2}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{c_1^3\tau^3}{2}\right\}\right.\\
&\left.+3\left\{\frac{\tau}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{3c_1^2\tau^2}{4}\right\}^2\right|\\
=&\frac{\tau^2}{12}\left|\left(\frac{3c_1^4}{4} - \frac{3c_1^2}{4}\left(c_2 - \frac{c_1^2}{2}\right)\right)\tau + \frac{c_1^4}{2} - c_1c_3 + c_1^2c_2 + \frac{3}{4}\left(c_2 - \frac{c_1^2}{2}\right)^2\right|.
\end{aligned} \tag{2.9}$$

It is known (1.2), that

$$\forall n \in \mathbb{N},\ \tau = \frac{\tau^n}{u_n} - x_n, \quad x_n = \frac{u_{n-1}}{u_n}, \quad \lim_{n\to\infty}\frac{u_{n-1}}{u_n} = |\tau| \approx 0.618. \tag{2.10}$$

Applying (2.10) gives

$$\begin{aligned}
|a_2a_4 - a_3^2| \;=\; &\frac{\tau^2}{12}\left|\left(\frac{3c_1^4}{4} - \frac{3c_1^2}{4}\left(c_2 - \frac{c_1^2}{2}\right)\right)\frac{\tau^n}{u_n} + c_1(c_1c_2 - c_3)\right.\\
&\left.+\frac{3}{4}c_2\left(c_2 - \frac{c_1^2}{2}\right) + \frac{3}{8}(2x_n - 1)c_1^2\left(c_2 - \frac{c_1^2}{2}\right) + \frac{2 - 3x_n}{4}c_1^4\right|.
\end{aligned}$$

Now, applying the triangle inequality, (1.4), (1.5) and (1.6) gives

$$\begin{aligned} |a_2a_4 - a_3^2| &\le \frac{\tau^2}{12}\left|\frac{3c_1^4}{4} - \frac{3c_1^2}{4}\left(c_2 - \frac{c_1^2}{2}\right)\right|\frac{|\tau|^n}{u_n} \\ &+ \frac{\tau^2}{12}\left\{|c_1||c_1c_2 - c_3| + \frac{3}{4}|c_2|\left|c_2 - \frac{c_1^2}{2}\right|\right. \\ &\left.+ \frac{3}{8}(2x_n - 1)|c_1^2|\left|c_2 - \frac{c_1^2}{2}\right| + \frac{2 - 3x_n}{4}|c_1|^4\right\} \\ &\le \frac{\tau^2}{12}\left|\frac{3c_1^4}{4} - \frac{3c_1^2}{4}\left(c_2 - \frac{c_1^2}{2}\right)\right|\frac{|\tau|^n}{u_n} \\ &+ \frac{\tau^2}{12}\left\{2|c_1| + \frac{3}{2}\left(2 - \frac{|c_1|^2}{2}\right)\right. \\ &\left.+ \frac{3}{8}(2x_n - 1)|c_1|^2\left(2 - \frac{|c_1|^2}{2}\right) + \frac{2 - 3x_n}{4}|c_1|^4\right\}, \end{aligned}$$

because by (2.10), we have $x_n \to 0.618$ so $2x_n - 1 > 0$, $2 - 3x_n > 0$ for sufficiently large n. So, in above calculation, in the last line, we have got a function of variable $|c_1| =: y \in [0, 2]$ and after elementary calculations we can get that

$$\max_{y\in[0,2]}\left\{2y + \frac{3}{2}\left(2 - \frac{y^2}{2}\right) + \frac{3}{8}(2x_n - 1)y^2\left(2 - \frac{y^2}{2}\right) + \frac{2 - 3x_n}{4}y^4\right\} = 12 - 12x_n \text{ at } y = 2. \tag{2.11}$$

Furthermore, it is clear that

$$\lim_{n\to\infty}\left|\frac{3c_1^4}{4} - \frac{3c_1^2}{4}\left(c_2 - \frac{c_1^2}{2}\right)\right|\frac{|\tau|^n}{u_n} = 0$$

and (2.10), (2.11) give

$$\begin{aligned} &\lim_{n\to\infty}\left[\max_{y\in[0,2]}\left\{2y + \frac{3}{2}\left(2 - \frac{y^2}{2}\right) + \frac{3}{8}(2x_n - 1)y^2\left(2 - \frac{y^2}{2}\right) + \frac{2 - 3x_n}{4}y^4\right\}\right] \\ &= 12 - 12|\tau| = 12\tau^2, \end{aligned}$$

so we have

$$|a_2a_4 - a_3^2| \le 0 + \frac{\tau^2}{12}12\tau^2 = \tau^4.$$

If we take in (2.2)

$$h(z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + \dots,$$

then putting $c_1 = c_2 = c_3 = 2$ in (2.9) gives

$$|a_2a_4 - a_3^2| = \frac{\tau^2}{12}|12\tau + 12| = \frac{\tau^2}{12}\left|12\tau^2\right| = \tau^4.$$

and it shows that (2.1) is sharp. It completes the proof. □

Theorem 2.2. *If $f(z) = z + a_2z^2 + \dots$ belongs to $\mathcal{SL}$, then*

$$|a_2a_3 - a_4| \le |\tau|^3. \tag{2.12}$$

The bound is sharp.

Proof. Let $f \in \mathcal{SL}$ and $p \in \mathcal{P}$ where $p(z) = 1 + p_1z + p_2z^2 + \cdots$. From (2.6), (2.7), (2.8) and

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + \cdots = 1 + p_1z + p_2z^2 + \cdots$$

we have

$$a_2a_3 - a_4 = \frac{1}{3}(p_1^3 - p_3).$$

So we obtain

$$\begin{aligned} |a_2a_3 - a_4| &= \frac{1}{3}\left|p_1^3 - p_3\right| \\ =&\frac{1}{3}\left|\frac{c_1^3}{8}\tau^3 - \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)\tau - \frac{3}{2}c_1\left(c_2 - \frac{c_1^2}{2}\right)\tau^2 - \frac{1}{2}c_1^3\tau^3\right| \\ =&\frac{1}{3}\left|\left\{\frac{1}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}\left(c_1c_2 - c_3\right) - \frac{7}{4}c_1c_2\right\}\tau + \frac{3}{8}c_1^3 - \frac{3}{2}c_1c_2\right|. \end{aligned} \tag{2.13}$$

Applying (2.10), we have

$$\begin{aligned} |a_2a_3 - a_4| &= \frac{1}{3}\left|\left\{\frac{1}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}\left(c_1c_2 - c_3\right) - \frac{7}{4}c_1c_2\right\}\frac{\tau^n}{u_n}\right. \\ &\left. -\frac{1}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right)x_n - \frac{1}{2}\left(c_1c_2 - c_3\right)x_n + \frac{7}{4}c_1c_2x_n + \frac{3}{8}c_1^3 - \frac{3}{2}c_1c_2\right|. \\ =&\frac{1}{3}\left|\left\{\frac{1}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}\left(c_1c_2 - c_3\right) - \frac{7}{4}c_1c_2\right\}\frac{\tau^n}{u_n}\right. \\ &+\frac{1}{2}\left(c_3 - 2c_1c_2 + c_1^3\right)x_n + \frac{3}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right)x_n \\ &\left. +\frac{5}{4}c_1c_2x_n - \frac{3}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) - \frac{3}{4}c_1c_2\right| \end{aligned} \tag{2.14}$$

Now, applying the triangle inequality, (1.4), (1.5),(1.6) and (1.7) gives

$$\begin{aligned}
|a_2a_3 - a_4| &\le \frac{1}{3}\left|\left\{\frac{1}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}(c_1c_2 - c_3) - \frac{7}{4}c_1c_2\right\}\right|\frac{|\tau^n|}{u_n} \\
&+ \frac{1}{2}\left|c_3 - 2c_1c_2 + c_1^3\right| x_n + \frac{|3x_n - 3|}{4}|c_1|\left|c_2 - \frac{c_1^2}{2}\right| + \frac{|5x_n - 3|}{4}|c_1||c_2| \\
&\le \frac{1}{3}\left|\left\{\frac{1}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}(c_1c_2 - c_3) - \frac{7}{4}c_1c_2\right\}\right|\frac{|\tau^n|}{u_n} \\
&+ \frac{1}{2}\left|c_3 - 2c_1c_2 + c_1^3\right| x_n + \frac{|3x_n - 3|}{4}|c_1|\left(2 - \frac{|c_1|^2}{2}\right) + \frac{|5x_n - 3|}{4}|c_1||c_2| \\
&\le \frac{1}{3}\left|\left\{\frac{1}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}(c_1c_2 - c_3) - \frac{7}{4}c_1c_2\right\}\right|\frac{|\tau^n|}{u_n} \\
&+ x_n + x_n|c_1| - \frac{3 - 3x_n}{8}|c_1|^3, \qquad (2.15)
\end{aligned}$$

because by (2.10), we have $x_n \to 0.618$ so $3x_n - 3 < 0$, $5x_n - 3 > 0$ for sufficiently large n. If we put $|c_1| =: y \in [0,2]$ then and after elementary calculations we can get that $h(y) = x_n + x_ny - (3 - 3x_n)y^3/3$ increases in $[0,2]$. Therefore,

$$\max_{y\in[0,2]}\{h(y)\} = \max_{y\in[0,2]}\left\{x_n + x_ny - \frac{3 - 3x_n}{8}y^3\right\} = 6x_n - 3 \quad \text{at} \quad y = 2.$$

Because

$$\lim_{n\to\infty}\left|\frac{1}{4}c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{2}(c_1c_2 - c_3) - \frac{7}{4}c_1c_2\right|\frac{|\tau^n|}{u_n} = 0$$

and by (2.10)

$$\lim_{n\to\infty}\left[\max_{y\in[0,2]}\left\{x_n + x_ny - \frac{3 - 3x_n}{8}y^3\right\}\right] = 6|\tau| - 3 = -3(2\tau + 1) = -3\tau^3 = 3|\tau|^3,$$

we have

$$|a_2a_3 - a_4| \le 0 + \frac{3|\tau|^3}{3} = |\tau|^3.$$

If we take in (2.2)

$$h(z) = \frac{1 + z}{1 - z} = 1 + 2z + 2z^2 + \dots,$$

then putting $c_1 = c_2 = c_3 = 2$ in (2.13) gives

$$|a_2a_3 - a_4| = |\tau|^3.$$

and it shows that (2.12) is sharp. It completes the proof. □

Now, we can obtain an upper bound for $|H_3(1)|$ in the class $\mathcal{SL}$ as follows:

Theorem 2.3. *If $f(z) = z + a_2z^2 + \dots$ belongs to $\mathcal{SL}$, then*

$$|H_3(1)| \le 20\tau^6. \qquad (2.16)$$

Proof. Let $f \in \mathcal{SL}$. By the definition of third Hankel determinant,

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

where $a_1 = 1$, we have

$$|H_3(1)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{2.17}$$

Considering Lemma 1.4, Lemma 1.5, Theorem 2.1 and Theorem 2.2 in (2.17), we obtain

$$\begin{aligned} |H_3(1)| &\le |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2| \\ &\le 2\tau^2\tau^4 + 3|\tau|^3|\tau|^3 + 5\tau^2 3\tau^2 \\ &= 20\tau^6. \end{aligned}$$

□

3. Concluding, Remarks and Observations

In our present article, we have obtained sharp estimates of the third Hankel determinant for the class $\mathcal{SL}$ of analytic functions related to shell-like curves connected with the Fibonacci numbers. Firstly, we have proved a conjecture given in [17] for sharp upper bound of second Hankel determinant. Secondly, we have obtained another sharp coefficient bound which will be used in the problem of finding the upper bound associated with the third Hankel determinant $H_3(1)$ for this class. Lastly, we have given an upper bound for functional $|H_3(1)|$ in the class $\mathcal{SL}$.

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