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Finite Blaschke Products and Decomposition

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Abstract: Let $B(z)$ be a finite Blaschke product of degree n. We consider the problem when a finite Blaschke product can be written as a composition of two nontrivial Blaschke products of lower degree related to the condition $B \circ M = B$ where M is a Möbius transformation from the unit disk onto itself.

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1. Introduction

It is known that a Möbius transformation from the unit disc $\mathbb{D}$ onto itself is of the following form:

$$M(z) = c\frac{z-\alpha}{1-\overline{\alpha}z}, \tag{1.1}$$

where $\alpha \in \mathbb{D}$ and c is a complex constant of modulus one (see [5] and [8]).

The rational function

$$B(z) = c\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}$$

is called a finite Blaschke product of degree n for the unit disc where $|c| = 1$ and $|a_k| < 1$, $1 \leq k \leq n$.

Blaschke products of the following form are called canonical Blaschke products:

$$B(z) = z\prod_{k=1}^{n-1}\frac{z-a_k}{1-\overline{a_k}z},\ |a_k| < 1 \text{ for } 1 \leq k \leq n-1. \tag{1.2}$$

It is well-known that every canonical Blaschke product B of degree n, is associated with a unique Poncelet curve (for more details see [2], [4] and [9]).

Decomposition of finite Blaschke products is an interesting matter studied by many researchers by the use of a point λ on the unit circle $\partial\mathbb{D}$ and the points $z_1, z_2, ..., z_n$ on the unit circle $\partial\mathbb{D}$ such that $B(z_1) = ... = B(z_n) = \lambda$. For example, using circles passing through the origin, it was given the determination of these points for the Blaschke products written as composition of two nontrivial Blaschke products of lower degree (see [11] and [12]). On the other hand, decomposition of finite Blaschke products is related to the condition $B \circ M = B$ where M is a Möbius transformation of the form (1.1) and different from the identity (see [1] and [8]). Some of the recent studies about decomposibility of finite Blaschke products can be found in [3].

In this paper we consider the relationship between the following two questions for a given canonical finite Blaschke product:

$Q1$) Is there a Möbius transformation M such that $B \circ M = B$ and M is different from the identity?

$Q2$) Can B be written as a composition $B = B_2 \circ B_1$ of two finite Blaschke products of lower degree?

Also, the above problems have been considered in details due to group theory in [7]. In the present study, we focus on a special class of finite Blaschke products (canonical finite Blaschke products).

In Section 2, we recall some known theorems about these questions. In Section 3 we give some theorems and examples related to the above two questions.

2. Preliminaries

In this section we give some information about decomposition of finite Blaschke products written as $B \circ M = B$ where M is a Möbius transformation different from the identity. In [8], it was proved that the set of continuous functions M from the unit disc into the unit disc such that $B \circ M = B$ is a cyclic group if B is a finite Blaschke product. In [1], the condition $B \circ M = B$ was used in the following theorem.

Theorem 2.1. (*See [1], Theorem* 3.1 *on page* 335) *Let B be a finite Blaschke product of degree n. Suppose $M \neq I$ is holomorphic from $\mathbb{D}$ into $\mathbb{D}$ such that $B \circ M = B$. Then*:

(*i*) *M is a Möbius transformation,*

(*ii*) *There is a positive integer $k \geq 2$ such that the iterates, $M, ...M^{k-1}$ are all distinct but $M^k = I$.*

(*iii*) *k divides n.*

(*iv*) *There is a $\gamma \in \mathbb{D}$ such that $M(\gamma) = \gamma$.*

(*v*) *B can be written as a composition* $B = B_2 \circ B_1$ *of finite Blaschke products with the degree* $B_1 = k$ *and the degree of* $B_2 = n/k$. B_1 *may be taken to be*

$$B_1(z) = \left(\frac{z - \gamma}{1 - \overline{\gamma} z} \right)^k .$$

But the condition $B \circ M = B$ is not necessary for a decomposition of finite Blaschke products (see [1] for more details).

It follows from Theorem 2.1 that if a finite Blaschke product B can be written as $B \circ M = B$, then B can be decomposed into a composition of two finite Blaschke products of lower order. However the following theorem gives necessary and sufficient conditions for the question $Q1$.

Theorem 2.2. (*See [8], Proposition* 4.1 *on page* 202) *Let B be finite Blaschke product of degree* $n \geq 1$, $B(z) = \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z}$ *with* $a_k \in \mathbb{D}$ *for* $1 \leq k \leq n$. *Let M be a Möbius transformation from* $\mathbb{D}$ *onto* $\mathbb{D}$. *The following assertions are equivalent*:

(*i*) $(B \circ M)(z) = B(z), z \in \mathbb{C}, |z| \leq 1.$

(*ii*) $M(\{a_1, a_2, ..., a_n\}) = \{a_1, a_2, ..., a_n\}$ *and there exists* $z_0 \in \overline{\mathbb{D}} \backslash \{a_1, a_2, ..., a_n\}$ *such that* $(B \circ M)(z_0) = B(z_0)$.

Using the following proposition given in [8], we know how to construct a finite Blaschke product of degree $n \geq 1$ satisfying the condition $B \circ M = B$ where M is a Möbius transformation from $\mathbb{D}$ into $\mathbb{D}$ different from identity.

Proposition 2.1. (*See [8], Proposition* 4.2 *on page* 203) *Let n be a positive integer and let M be a Möbius transformation from* $\mathbb{D}$ *into* $\mathbb{D}$ *such that* $M^n(0) = 0$ *and* $\{0, M(0), ..., M^{n-1}(0)\}$ *is a set of n distinct points in* $\mathbb{D}$. *Consider the finite Blaschke product* $B(z) = z \prod_{k=1}^{n-1} \frac{z - M^k(0)}{1 - \overline{M^k(0)} z}$. *Then the group G of the invariants of B is generated by M.*

From [10], we know the following theorem and we will use this theorem in the next chapter.

Theorem 2.3. *Let*

$$A(z) = \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z} \text{ and } B(z) = \prod_{k=1}^{n} \frac{z - b_k}{1 - \overline{b_k} z}$$

with a_k *and* $b_k \in \mathbb{D} = \{ |z| < 1 \}$ *for* $k = 1, 2, ..., n$. *Suppose that* $A(\lambda_k) = B(\lambda_k)$ *for n distinct points* $\lambda_1, ..., \lambda_n$ *in* $\mathbb{D}$. *Then* $A \equiv B$.

3. Blaschke products of degree n

Let B be a canonical Blaschke product of degree n and following [8], let $Z(B)$ denotes the set of the elements $z \in \mathbb{D}$ such that $B(z) = 0$. In this section we consider the relationship between the questions (Q1) and (Q2).

Now we give the following theorem for the Blaschke products of degree n.

Theorem 3.1. *Let $M(z) = c\frac{z-\alpha}{1-\overline{\alpha}z}$ be a Möbius transformation different from the identity from the unit disc into itself and $B(z) = z\prod\limits_{k=1}^{n-1}\frac{z-a_k}{1-\overline{a_k}z}$ be a canonical Blaschke product of degree n. Then $B \circ M = B$ if and only if $M(z) = c\frac{z-a_j}{1-\overline{a_j}z}$ with $|a_j| = |a_l|$ for some a_j, a_l $(0 \leq j, l \leq n-1, j \neq l)$, $c = -\frac{a_j}{a_l}$ and the equation*

$$M^{n-1}(0) - a_i = 0, \ (1 \leq i \leq n-1) \tag{3.1}$$

is satisfied by the non-zero zeros of B.

Proof. Necessity: Let $M(z) = c\frac{z-\alpha}{1-\overline{\alpha}z}$ be a Möbius transformation different from the identity from the unit disc into itself, $B(z) = z\prod\limits_{k=1}^{n-1}\frac{z-a_k}{1-\overline{a_k}z}$ be a canonical Blaschke product of degree n and $B \circ M = B$. From Proposition 2.1, we know $Z(B) = \{0, M(0), ..., M^{n-1}(0)\}$ and $M^n(0) = 0$. Without loss of generality, let us take $a_1 = M(0)$. Then we find

$$a_1 = -c\alpha \tag{3.2}$$

Let $a_j = M^j(0)$ $(2 \leq j \leq n-1)$ and then we find the following equations:

$$a_2 = M^2(0) = -\frac{c\alpha(1+c)}{1+c|\alpha|^2}, \ a_3 = M^3(0)$$

$$\vdots$$

$$a_{n-1} = M^{n-1}(0). \tag{3.3}$$

By Theorem 2.1, then it should be $M^n(0) = 0$. Using the equation (3.3) we have

$$M^n(0) = M\left(M^{n-1}(0)\right) = M(a_{n-1}) = 0$$

and so we get

$$a_{n-1} = \alpha.$$

By the equation (3.2) we find $c = -\frac{a_1}{a_{n-1}}$ and hence $|a_1| = |a_{n-1}|$. If we take $a_1 = a_j$ and $a_{n-1} = a_l$ then the proof follows.

Sufficiency: For the points 0 and a_k $(1 \leq k \leq n-1)$ in $\mathbb{D}$, we have

$$(B \circ M)(0) = B(0) \ \text{ and } \ (B \circ M)(a_k) = B(a_k),$$

by the equation (3.1). Then, by Theorem 2.3 we obtain

$$B \circ M \equiv B.$$

□

Notice that if all $a_k = 0$, then we know that $M(z) = e^{\frac{2\pi i}{n}}$ (see [8] on page 202).

From [8], we know the following corollary for the Blaschke products of degree 3.

Corollary 3.1. (*See [8], page* 205) *Let G be the cyclic group which is composed of the transformations M such that $B \circ M = B$. Then we have the following assertions*:

- (*i*) *If $B(z) = z^3$, G is generated by $M(z) = e^{2i\pi/3}z$, $z \in \overline{\mathbb{D}}$.*
- (*ii*) *If $Z(B)$ contains a non-zero point in $\mathbb{D}$, $B(z) = z\frac{z-a_1}{1-\overline{a_1}z}\frac{z+\overline{c}a_1}{1+c\overline{a_1}z}$ where $\alpha \in \mathbb{D}\backslash\{0\}$ and where $c + \overline{c} = -1 - |a_1|^2$. In this case the group G is generated by $M(z) = c\frac{z+\overline{c}a_1}{1+c\overline{a_1}z}$.*

However, as an application of Theorem 3.1, we give the following corollary in our form for degree 3.

Corollary 3.2. *Let $M(z) = c\frac{z-a_1}{1-\overline{a_1}z}$ be a Möbius transformation different from the identity from the unit disc onto itself and $B(z) = z\frac{z-a}{1-\overline{a}z}\frac{z-b}{1-\overline{b}z}$ be a Blaschke product of degree 3. Then $B \circ M = B$ if and only if $M(z) = c\frac{z-b}{1-\overline{b}z}$ with $|a| = |b|$ and some c where c is a root of the equation $c^2 + c(1 + |a|^2) + 1 = 0$ with $|c| = 1$.*

As an other application of Theorem 3.1, a similar corollary can be given for the Blaschke products of prime degrees. We give the following corollaries and examples for degree 5 and 7.

Corollary 3.3. *Let $M(z) = c\frac{z-\alpha}{1-\overline{\alpha}z}$ be a Möbius transformation different from the identity from the unit disc onto itself and $B(z) = z\prod\limits_{k=1}^{4}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree 5. Then $B \circ M = B$ if and only if $M(z) = c\frac{z-a_l}{1-\overline{a_l}z}$ with $|a_j| = |a_l|$ for some $(0 < j, l \leq 4)$, $c = -\frac{a_j}{a_l}$ and the equation*

$$4c^2a_l|a_l|^2 + 3ca_l|a_l|^2 + 3c^3a_l|a_l|^2 + c^4a_l + c^3a_l + c^2a_l + ca_l + a_l + c^2a_l|a_l|^4 = 0, \quad (3.4)$$

is satisfied by the non-zero zeros of B.

Example 3.1. Let $B(z) = z\prod\limits_{k=1}^{4}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree 5. From Proposition 2.1, we know that $Z(B) = \{0, M(0), ..., M^4(0)\}$, so we can take

$$\begin{aligned} a_1 &= M(0), \\ a_2 &= M^2(0), \\ a_3 &= M^3(0), \\ a_4 &= M^4(0). \end{aligned} \quad (3.5)$$

Let $a_l = \frac{1}{2}$. Using the equation (3.4), we obtain $c = -0.856763 - i0.515711$ and

$$M(z) = \frac{(0.856763 + i0.515711)(1-2z)}{2-z}.$$

Using the equation (3.5), we find

$$\begin{aligned} a_1 &= 0.428381 + i0.257855, \\ a_2 &= 0.278236 - i0.188486, \\ a_3 &= 0.141178 + 0.304977i, \\ a_4 &= 0.5. \end{aligned}$$

Then we find $(B \circ M)(z) = B(z)$ for the points $z \in \mathbb{D}$.

Corollary 3.4. *Let $M(z) = c\frac{z-\alpha}{1-\overline{\alpha}z}$ be a Möbius transformation different from the identity from the unit disc into itself and $B(z) = z\prod_{k=1}^{6}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree 7. Then $B \circ M = B$ if and only if $M(z) = c\frac{z-a_l}{1-\overline{a_l}z}$ with $|a_j| = |a_l|$ for some $(0 < j, l \leq 6)$, $c = -\frac{a_j}{a_l}$ and the equation*

$$\begin{aligned} & a_l + 5ca_l\,|a_l|^2 + 8c^2a_l\,|a_l|^2 + 9c^3a_l\,|a_l|^2 + 8c^4a_l\,|a_l|^2 \\ & +5c^5a_l\,|a_l|^2 + 6c^2a_l\,|a_l|^4 + 9c^3a_l\,|a_l|^4 + 6c^4a_l\,|a_l|^4 \\ & +c^3a_l\,|a_l|^6 + ca_l + c^2a_l + c^3a_l + c^4a_l + c^5a_l + c^6a_l = 0 \end{aligned} \tag{3.6}$$

is satisfied by the non-zero zeros of B.

Example 3.2. Let $B(z) = z\prod_{k=1}^{6}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree 7. From Proposition 2.1, we know that $Z(B) = \{0, M(0), ..., M^6(0)\}$, so we can take

$$\begin{aligned} a_1 &= M(0), \\ a_2 &= M^2(0), \\ a_3 &= M^3(0), \\ a_4 &= M^4(0), \\ a_5 &= M^5(0), \\ a_6 &= M^6(0). \end{aligned} \tag{3.7}$$

Let $a_l = \frac{1}{2}$. Using the equation (3.6), we obtain $c = 0.217617 - i0.976034$ and

$$M(z) = \frac{-(0.217617 - i0.976034)(2z-1)}{2-z}.$$

Using the equation (3.7), we find

$$\begin{aligned} a_1 &= -0.108809 + i0.488017, \\ a_2 &= 163605 + i0.702141, \\ a_3 &= 0.40682 + i0.679542, \\ a_4 &= 0.574725 + i0.54495, \\ a_5 &= 0.64971 + i0.312482, \\ a_6 &= 0.5. \end{aligned}$$

Then we obtain $(B \circ M)(z) = B(z)$ for the points $z \in \mathbb{D}$.

Now we consider the canonical Blaschke products of degree 4. At first, from [8], we can give the following corollary for a Blaschke product B of degree 4 .

Corollary 3.5. (*See [8], page* 204) *Let G be a cyclic group which is composed of the transformations M such that $B \circ M = B$. Then we have the following assertions*:

(*i*) *If $B(z) = z^4$, G is generated by $M(z) = iz$, $z \in \overline{\mathbb{D}}$.*

(*ii*) *If $Z(B)$ contains a non-zero point in $\mathbb{D}$, there are two cases*:

(*a*)

$$B(z) = z\frac{z-a_1}{1-\overline{a_1}z}\frac{z-a_2}{1-\overline{a_2}z}\frac{z-\frac{a_1-a_2}{1-\overline{a_1}a_2}}{1-z\left(\frac{\overline{a_1}-\overline{a_2}}{1-a_1\overline{a_2}}\right)},$$

where a_1 or a_2 is non-equal to 0. *In this case, $M(z) = -\frac{z-a_1}{1-\overline{a_1}z}$ and thus M does not generate G since the degree of M is equal to* 2.

(*b*)

$$B(z) = z\frac{z-a_1}{1-\overline{a_1}z}\frac{z-M(a_1)}{1-\overline{M(a_1)}z}\frac{z-M^2(a_1)}{1-\overline{M^2(a_1)}z},$$

with $a_1 \in \mathbb{D}$. In this case $M(z) = c\frac{z+\overline{c}a_1}{1+c\overline{a_1}z}$ generates G with $|c| = 1$ and $c + \overline{c} = -2|a_1|^2$.

In this case there is a nice relation between decomposition of the finite Blaschke products B of order 4 and the Poncelet curves associated with them. From [6] and [13], we know the following theorem.

Theorem 3.2. *For any Blaschke product B of order* 4, *B is a composition of two Blaschke products of degree* 2, *that is $B(z) = (f \circ g)(z)$ where $f(z) = z\frac{z-\alpha}{1-\overline{\alpha}z}$, $g(z) = z\frac{z-\beta}{1-\overline{\beta}z}$, $\alpha = -a_1a_2$ and $\beta = \frac{a_1+a_2-a_1a_2(\overline{a_1}+\overline{a_2})}{1-|a_1a_2|^2}$, if and only if the Poncelet curve E of this Blaschke product is an ellipse with the equation*

$$E : |z-a_1| + |z-a_2| = |1-\overline{a_1}a_2|\sqrt{\frac{|a_1|^2+|a_2|^2-2}{|a_1|^2|a_2|^2-1}}.$$

It is also known that the decomposition of some Blaschke products B of degree 4 is linked with the case that Poncelet curve of this Blaschke product is an ellipse with a nice geometric property.

Theorem 3.3. (*See [13], Theorem* 5.2 *on page* 103) *Let a_1, a_2 and a_3 be three distinct nonzero complex numbers with $|a_i| < 1$ for $1 \leq i \leq 3$ and $B(z) = z\prod_{i=1}^{3} \frac{z-a_i}{1-\overline{a_i}z}$ be a Blaschke product of degree 4 with the condition that one of its zeros, say a_1,satisfies the following equation:*

$$a_1 + \overline{a_1}a_2a_3 = a_2 + a_3.$$

Then the Poncelet curve associated with B is the ellipse E with the equation

$$E : |z - a_2| + |z - a_3| = |1 - \overline{a_2}a_3| \sqrt{\frac{|a_2|^2 + |a_3|^2 - 2}{|a_2|^2 |a_3|^2 - 1}}.$$

Let $B(z)$ be given as in the statement of Theorem 3.3. For any $\lambda \in \partial\mathbb{D}$, let z_1, z_2, z_3 and z_4 be the four distinct points satisfying $B(z_1) = B(z_2) = B(z_3) = B(z_4) = \lambda$. Then the Poncelet curve associated with B is an ellipse E with foci a_2 and a_3 and the lines joining z_1, z_3 and z_2, z_4 pass through the point a_1.

Example 3.3. Let $a_1 = \frac{2}{3}, a_2 = \frac{1}{2} - i\frac{1}{2}, a_3 = \frac{1}{2} + i\frac{1}{2}$ and $B(z) = z\prod_{i=1}^{3} \frac{z-a_i}{1-\overline{a_i}z}$. The Poncelet curve associated with B is an ellipse with foci a_2 and a_3 (see Figure 1).

However decomposition of a Blaschke product B is not always linked with the Poncelet curve of the Blaschke product, as we will see in the following theorem.

Theorem 3.4. (*See [11], Theorem* 4.2 *in page* 69) *Let $a_1, a_2, ..., a_{2n-1}$ be $2n - 1$ distinct nonzero complex numbers with $|a_k| < 1$ for $1 \leq k \leq 2n - 1$ and $B(z) = z\prod_{k=1}^{2n-1} \frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree $2n$ with the condition that one of its zeros, say a_1, satisfies the following equations*:

$$\begin{aligned} a_1 + \overline{a_1}a_2a_3 &= a_2 + a_3. \\ a_1 + \overline{a_1}a_4a_5 &= a_4 + a_5. \\ &\cdots \\ a_1 + \overline{a_1}a_{2n-2}a_{2n-1} &= a_{2n-2} + a_{2n-1}. \end{aligned}$$

(*i*) *If L is any line through the point a_1, then for the points z_1 and z_2 at which L intersects $\partial\mathbb{D}$, we have $B(z_1) = B(z_2)$.*

(*ii*) *The unit circle $\partial\mathbb{D}$ and any circle through the points 0 and $\frac{1}{\overline{a_1}}$ have exactly two distinct intersection points z_1 and z_2. Then we have $B(z_1) = B(z_2)$ for these intersection points.*

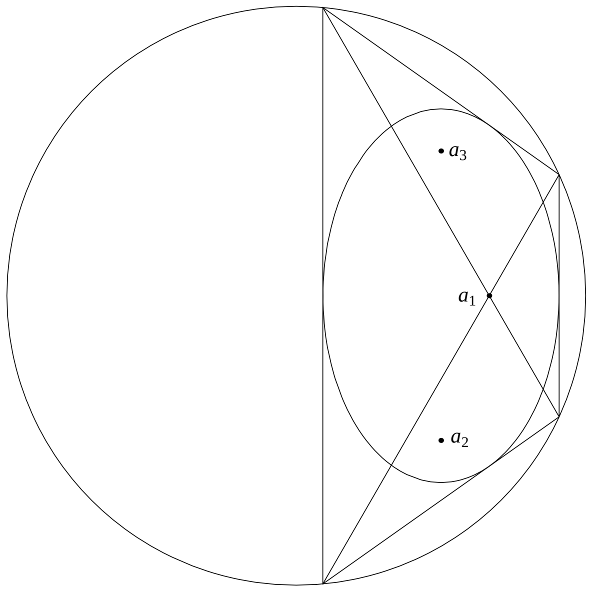

Figure 1:

From the proof of Theorem 3.4, we know that Blaschke product B can be written as $B(z) = B_2 \circ B_1(z)$ where

$$B_1(z) = z\frac{z - a_1}{1 - \overline{a}_1 z} \text{ and } B_2(z) = z\frac{(z + a_2a_3)(z + a_4a_5)...(z + a_{2n-2}a_{2n-1})}{(1 + \overline{a_2a_3}z)(1 + \overline{a_4a_5}z)...(1 + \overline{a_{2n-2}a_{2n-1}}z)}.$$

Now we investigate under what conditions $B \circ M = B$ such that M is different from the identity for the Blaschke product given in Theorem 3.4. For $n = 2$, we can give the following theorem.

Theorem 3.5. *Let a_1, a_2 and a_3 be three distinct nonzero complex numbers with $|a_k| < 1$ for $1 \leq k \leq 3$ and $B(z) = z\prod_{k=1}^{3} \frac{z-a_k}{1-\overline{a}_k z}$ be a Blaschke product of degree 4 with the condition that one of its zeros, say a_1, satisfies the following equation*:

$$a_1 + \overline{a_1}a_2a_3 = a_2 + a_3.$$

(i) If $M(z) = -\frac{z-a_1}{1-\overline{a}_1 z}$ then we have $B = B_2 \circ B_1$ and $B = B \circ M$ only when $a_3 = \frac{a_1-a_2}{1-\overline{a}_1 a_2}$.

(ii) Let $M(z) = c\frac{z+\overline{c}a_1}{1+c\overline{a}_1 z}$ with the conditions $c + \overline{c} = -2\,|a_1|^2$, $a_2 = M(a_1)$ and $a_3 = M^2(a_1)$. If a_1 and c with $|c| = 1$ satisfy the following equation

$$a_1c + 2a_1c^2\,|a_1|^2 + a_1c^3\,|a_1|^2 + a_1 - a_1\,|a_1|^2 = 0, \tag{3.8}$$

then we have $B = B_2 \circ B_1$ and $B = B \circ M$.

Proof. We use the equation $a_1+\overline{a_1}a_2a_3=a_2+a_3$, Theorem 3.4 and Corollary 3.5.

(i) Let $M(z)=-\frac{z-a_1}{1-\overline{a}_1 z}$. Then by Corollary 3.5, the condition $B=B\circ M$ implies $a_3=\frac{a_1-a_2}{1-\overline{a}_1 a_2}$.

(ii) Let $M(z)=c\frac{z+\overline{c}a_1}{1+c\overline{a}_1 z}$. From Corollary 3.5, it should be $a_2=M(a_1)$, $a_3=M^2(a_1)$ and $c+\overline{c}=-2\left|a_1\right|^2$. Then we obtain

$$a_2=\frac{a_1(1+c)}{1+c\left|a_1\right|^2} \text{ and } a_3=\frac{a_1(1-\left|a_1\right|^2)}{1+2c\left|a_1\right|^2+c^2\left|a_1\right|^2}.$$

If we substitute these values of a_2 and a_3 in the equation $a_1+\overline{a_1}a_2a_3=a_2+a_3$, we have the following equation

$$a_1c+2a_1c^2\left|a_1\right|^2+a_1c^3\left|a_1\right|^2+a_1-a_1\left|a_1\right|^2=0.$$

Also in both cases we know that B has a decomposition as $B=B_2\circ B_1$ by Theorem 3.3. Thus the proof is completed. □

Now, we give two examples for the both cases of the above theorem.

Example 3.4. Let B be a Blaschke product and M be a Möbius transformation of the following forms:

$$B(z)=z\frac{z-a_1}{1-\overline{a}_1 z}\frac{z-a_2}{1-\overline{a}_2 z}\frac{z-\left(\frac{a_1-a_2}{1-\overline{a}_1 a_2}\right)}{1-z\left(\frac{\overline{a}_1-\overline{a}_2}{1-a_1\overline{a}_2}\right)} \text{ and } M(z)=\frac{-z+a_1}{1-\overline{a}_1 z}.$$

For $a_1=\frac{1}{2}$ and $a_2=\frac{1}{2}-\frac{i}{2}$ we obtain

$$B(z)=\frac{z(z-\frac{1}{2})(z-\frac{1}{2}+\frac{i}{2})(z-\frac{1}{5}-\frac{3i}{5})}{(-\frac{1}{2}z+1)(1-z(\frac{1}{2}+\frac{i}{2}))(1-z(\frac{1}{5}-\frac{3i}{5}))} \text{ and } M(z)=\frac{-z+\frac{1}{2}}{1-\frac{1}{2}z}.$$

Then we find $(B\circ M)(z)=B(z)$ and $B(z)=(B_2\circ B_1)(z)$ for the points $z\in\mathbb{D}$.

Example 3.5. Let B be a Blaschke product and M be a Möbius transformation of the following forms:

$$B(z)=z\frac{z-a_1}{1-\overline{a_1}z}\frac{z-M(a_1)}{1-\overline{M(a_1)}z}\frac{z-M^2(a_1)}{1-\overline{M^2(a_1)}z} \text{ and } M(z)=c\frac{z+\overline{c}a_1}{1+c\overline{a_1}z}.$$

For $a_1=\frac{2}{3}$, solving the equation $a_1c+2a_1c^2\left|a_1\right|^2+a_1c^3\left|a_1\right|^2+a_1-a_1\left|a_1\right|^2=0$ we find $c=-1$. Then we have B and M of the following forms:

$$B(z)=\frac{z^2(-\frac{2}{3}+z)^2}{(1-\frac{2}{3}z)^2} \text{ and } M(z)=-\frac{-\frac{2}{3}+z}{1-\frac{2}{3}z}$$

Then we find $(B\circ M)(z)=B(z)$ and $B(z)=(B_2\circ B_1)(z)$ for the points $z\in\mathbb{D}$.

From the above discussions, we can say that decomposition of a finite Blaschke product B is linked with its zeros. But for a finite Blaschke product B of degree 4, this case is also linked with the Poncelet curve of B.

Using Theorem 3.1 and Theorem 3.4, for the Blaschke product of degree $2n$, we give the following result.

Corollary 3.6. *Let $a_1, a_2, ..., a_{2n-1}$ be $2n-1$ distinct nonzero complex numbers with $|a_k| < 1$ for $1 \le k \le 2n-1$ and $B(z) = z \prod_{k=1}^{2n-1} \frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke products of degree $2n$ with the condition that one of its zeros, say a_1, satisfies the following equations:*

$$\begin{aligned} a_1 + \overline{a_1} a_2 a_3 &= a_2 + a_3 \\ a_1 + \overline{a_1} a_4 a_5 &= a_4 + a_5 \\ &\cdots \\ a_1 + \overline{a_1} a_{2n-2} a_{2n-1} &= a_{2n-2} + a_{2n-1}. \end{aligned}$$

Let $M(z) = c\frac{z-a_{2n-1}}{1-z\overline{a_{2n-1}}}$ with the conditions $|c| = 1$, $M^{2n-1}(0) - a_{2n-1} = 0$, $a_1 = M(0), a_2 = M^2(0), ..., a_{2n-1} = M^{2n-1}(0)$. If a_{2n-1} and c satisfy following equations:

$$\begin{aligned} M(0) + \overline{M(0)} M^2(0) M^3(0) &= M^2(0) + M^3(0) \\ M(0) + \overline{M(0)} M^4(0) M^5(0) &= M^4(0) + M^5(0) \\ &\cdots \\ M(0) + \overline{M(0)} M^{2n-2}(0) M^{2n-1}(0) &= M^{2n-2}(0) + M^{2n-1}(0) \end{aligned}$$

Then we have $B = B_2 \circ B_1$ and $B \circ M = B$.

Proof. The proof is obvious from Theorem 3.1 and Theorem 3.4. □

Example 3.6. Let B be a Blaschke product and M be a Möbius transformation of the following forms:

$$B(z) = z\frac{z-M(0)}{1-\overline{M(0)}z}\frac{z-M^2(0)}{1-\overline{M^2(0)}z}\frac{z-M^3(0)}{1-\overline{M^3(0)}z}\frac{z-M^4(0)}{1-\overline{M^4(0)}z}\,\frac{z-M^5(0)}{1-\overline{M^5(0)}z}$$

and

$$M(z) = c\frac{z-a_5}{1-\overline{a_5}z}.$$

From Corollary 3.6, it should be

$$M(0) + \overline{M(0)} M^2(0) M^3(0) = M^2(0) + M^3(0) \tag{3.9}$$

and

$$M(0) + \overline{M(0)} M^4(0) M^5(0) = M^4(0) + M^5(0). \tag{3.10}$$

We know that $a_1 = M(0) = -ca_5, a_2 = M^2(0) = -ca_5\frac{(1+c)}{1+c|a_5|^2}, a_3 = M^3(0) = -ca_5\frac{\left(1+c+c^2+c|a_5|^2\right)}{1+2c|a_5|^2+c^2|a_5|^2}$ and $a_4 = M^4(0) = -ca_5\frac{(1+c)\left(1+c^2+2c|a_5|^2\right)}{1+c|a_5|^2\left(3+2c+c^2+c|a_5|^2\right)}$. Writing these values in the equations (3.9) and (3.10), we have

$$\begin{aligned} &ca_5 + 2c^2a_5 + c^3a_5 + c^3a_5|a_5|^2 + c^4a_5|a_5|^2 - 2c^3a_5|a_5|^4 \\ &-c^4a_5|a_5|^4 - ca_5|a_5|^2 - c^2a_5|a_5|^2 - c^2a_5|a_5|^4 = 0 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &a_5 - c^2a_5 - c^3a_5 - c^4a_5 + 3ca_5|a_5|^2 + c^2a_5|a_5|^4 + 2c^2a_5|a_5|^2 + c^4a_5|a_5|^2 \\ &+c^3a_5|a_5|^4 - ca_5|a_5|^2 - 2c^2a_5|a_5|^4 - a_5|a_5|^2 - 2ca_5|a_5|^4 = 0. \end{aligned} \tag{3.12}$$

For $a_5 = \frac{1}{2}$, solving the equations (3.11) and (3.12) we find $c = -1$. Then we have B and M of the following forms:

$$B(z) = z\frac{(2z-1)(z-0.5)^2\left(z-1.4803\times 10^{-16}\right)\left(z-7.40149\times 10^{-17}\right)}{(2-z)(1-0.5z)^2\left(1-1.4803\times 10^{-16}z\right)\left(1-7.40149\times 10^{-17}z\right)}$$

and

$$M(z) = \frac{2z-1}{z-2}.$$

Then for the points $z \in \mathbb{D}$, we find

$$(B\circ M)(z) = B(z) \text{ and } B(z) = (B_2\circ B_1)(z)$$

where

$$B_1(z) = z\frac{z-0.5}{1-0.5z} \text{ and } B_2(z) = z\frac{z+3.70074\times 10^{-17}}{1+3.70074\times 10^{-17}z}\frac{z+7.40149\times 10^{-17}}{1+7.40149\times 10^{-17}z}.$$

Corollary 3.7. *Let $a_1, a_2, ..., a_{3n-1}$ be $3n-1$ distinct nonzero complex numbers with $|a_k| < 1$ for $1 \le k \le 3n-1$ and $B(z) = z\prod_{k=1}^{3n-1}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke products of degree $3n$ with the condition that one of its zeros, say a_1, satisfies the following equations:*

$$\begin{aligned} a_1 + a_2 + a_3a_4a_5\overline{a_1a_2} &= a_3 + a_4 + a_5, \\ a_1a_2 + a_3a_4a_5\left(\overline{a_1}+\overline{a_2}\right) &= a_3a_4 + a_3a_5 + a_4a_5, \\ &\cdots \\ a_1 + a_2 + a_{3n-3}a_{3n-2}a_{3n-1}\overline{a_1a_2} &= a_{3n-3} + a_{3n-2} + a_{3n-1}, \\ a_1a_2 + a_{3n-3}a_{3n-2}a_{3n-1}\left(\overline{a_1}+\overline{a_2}\right) &= a_{3n-3}a_{3n-2} + a_{3n-3}a_{3n-1} + a_{3n-2}a_{3n-1}. \end{aligned}$$

Let $M(z) = c\frac{z-a_{3n-1}}{1-z\overline{a_{3n-1}}}$ with the conditions $|c| = 1$, $M^{3n-1}(0) - a_{3n-1} = 0$, $a_1 = M(0), a_2 = M^2(0), ..., a_{3n-1} = M^{3n-1}(0)$. If a_{3n-1} and c satisfy following *equations:*

$$\begin{aligned}
&M(0)+M^2(0)+M^3(0)\,M^4(0)\,M^5(0)\,\overline{M(0)\,M^2(0)}\\
&=M^3(0)+M^4(0)+M^5(0),\\
&M(0)\,M^2(0)+M^3(0)\,M^4(0)\,M^5(0)\left(\overline{M(0)}+\overline{M^2(0)}\right)\\
&=M^3(0)\,M^4(0)+M^3(0)\,M^5(0)+M^4(0)\,M^5(0),\\
&\ldots\\
&M(0)+M^2(0)+M^{3n-3}(0)\,M^{3n-2}(0)\,M^{3n-1}(0)\,\overline{M(0)\,M^2(0)}\\
&=M^{3n-3}(0)+M^{3n-2}(0)+M^{3n-1}(0),\\
&M(0)\,M^2(0)+M^{3n-3}(0)\,M^{3n-2}(0)\,M^{3n-1}(0)\left(\overline{M(0)}+\overline{M^2(0)}\right)\\
&=M^{3n-3}(0)\,M^{3n-2}(0)+M^{3n-3}(0)\,M^{3n-1}(0)+M^{3n-2}(0)\,M^{3n-1}(0).
\end{aligned}$$

Then we have $B=B_2\circ B_1$ and $B\circ M=B$.

Proof. Let $a_1,a_2,...,a_{3n-1}$ be $3n-1$ distinct nonzero complex numbers with $|a_k|<1$ for $1\leq k\leq 3n-1$ and $B(z)=z\prod\limits_{k=1}^{3n-1}\frac{z-a_k}{1-\overline{a_k}z}$ be a Blaschke product of degree $3n$ with the condition that two of its zeros, say a_1 and a_2, satisfies the following equations:

$$\begin{aligned}
a_1+a_2+a_3a_4a_5\overline{a_1a_2} &= a_3+a_4+a_5,\\
a_1a_2+a_3a_4a_5\left(\overline{a_1}+\overline{a_2}\right) &= a_3a_4+a_3a_5+a_4a_5,\\
a_1+a_2+a_6a_7a_8\overline{a_1a_2} &= a_6+a_7+a_8,\\
a_1a_2+a_6a_7a_8\left(\overline{a_1}+\overline{a_2}\right) &= a_6a_7+a_6a_8+a_7a_8,\\
&\ldots\\
a_1+a_2+a_{3n-3}a_{3n-2}a_{3n-1}\overline{a_1a_2} &= a_{3n-3}+a_{3n-2}+a_{3n-1},\\
a_1a_2+a_{3n-3}a_{3n-2}a_{3n-1}\left(\overline{a_1}+\overline{a_2}\right) &= a_{3n-3}a_{3n-2}+a_{3n-3}a_{3n-1}+a_{3n-2}a_{3n-1}.
\end{aligned}$$

By Theorem 4.4 on page 71 in [11], we know that B(z) can be written as a composition of two Blaschke products of degree 3 and n as $B(z)=(B_2\circ B_1)(z)$ where

$$B_1(z)=\frac{z(z-a_1)(z-a_2)}{(1-\overline{a_1}z)(1-\overline{a_2}z)}$$

and

$$B_2(z)=\frac{z(z-a_3a_4a_5)(z-a_6a_7a_8)\ldots(z-a_{3n-3}a_{3n-2}a_{3n-1})}{(1-\overline{a_3a_4a_5}z)(1-\overline{a_6a_7a_8}z)\ldots(1-\overline{a_{3n-3}a_{3n-2}a_{3n-1}}z)}$$

Then, the rest of the proof is clear from Theorem 3.1. □

Example 3.7. Let B be a Blaschke product and M be a Möbius transformation of the following forms:

$$B(z)=z\frac{z-M(0)}{1-\overline{M(0)}z}\,\frac{z-M^2(0)}{1-\overline{M^2(0)}z}\,\frac{z-M^3(0)}{1-\overline{M^3(0)}z}\,\frac{z-M^4(0)}{1-\overline{M^4(0)}z}\,\frac{z-M^5(0)}{1-\overline{M^5(0)}z}$$

and

$$M(z) = c\frac{z-a_5}{1-\overline{a_5}z}.$$

From Corollary 3.7, it should be

$$\begin{aligned} M(0) + M^2(0) + M^3(0)\, M^4(0)\, M^5(0)\, \overline{M(0)\, M^2(0)} \\ = M^3(0) + M^4(0) + M^5(0) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} M(0)\, M^2(0) + M^3(0)\, M^4(0)\, M^5(0) \left(\overline{M(0)} + \overline{M^2(0)}\right) \\ = M^3(0)\, M^4(0) + M^3(0)\, M^5(0) + M^4(0)\, M^5(0)\,. \end{aligned} \tag{3.14}$$

We know that $a_1 = M(0) = -ca_5, a_2 = M^2(0) = -ca_5\frac{(1+c)}{1+c|a_5|^2}, a_3 = M^3(0) = -ca_5\frac{\left(1+c+c^2+c|a_5|^2\right)}{1+2c|a_5|^2+c^2|a_5|^2}$ and $a_4 = M^4(0) = -ca_5\frac{(1+c)\left(1+c^2+2c|a_5|^2\right)}{1+c|a_5|^2\left(3+2c+c^2+c|a_5|^2\right)}$. Writing these values in the equations (3.13) and (3.14), we have

$$\begin{aligned} &-ca_5\left(1+c|a_5|^2\right)\left(1+2c|a_5|^2+c^2|a_5|^2\right)\left(1+c|a_5|^2\left(3+2c+c^2+c|a_5|^2\right)\right)\left(1+\overline{c}|a_5|^2\right) \\ &-ca_5(1+c)\left(1+2c|a_5|^2+c^2|a_5|^2\right)\left(1+c|a_5|^2\left(3+2c+c^2+c|a_5|^2\right)\right)\left(1+\overline{c}|a_5|^2\right) \\ &+a_5|a_5|^4\left(1+c+c^2+c|a_5|^2\right)(1+c)\left(1+c^2+2c|a_5|^2\right)(1+\overline{c})\left(1+c|a_5|^2\right) \\ &+ca_5\left(1+c+c^2+c|a_5|^2\right)\left(1+c|a_5|^2\right)\left(1+c|a_5|^2\left(3+2c+c^2+c|a_5|^2\right)\right)\left(1+\overline{c}|a_5|^2\right) \\ &+ca_5(1+c)\left(1+c^2+2c|a_5|^2\right)(1+c|a_5|)^2\left(1+2c|a_5|^2+c^2|a_5|^2\right)(1+\overline{c}|a_5|)^2 \\ &-a_5\left(1+c|a_5|^2\right)\left(1+2c|a_5|^2+c^2|a_5|^2\right)\left(1+c|a_5|^2\left(3+2c+c^2+c|a_5|^2\right)\right)\left(1+\overline{c}|a_5|^2\right)=0. \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &c^2a_5^2(1+c)\left(1+2c|a_5|^2+c^2|a_5|^2\right)\left(1+c|a_5|^2\left(3+2c+c^2+c|a_5|^2\right)\right)\left(1+\overline{c}|a_5|^2\right)\left(1+\overline{c}|a_5|^2\right) \\ &+c^2a_5^3\left(1+c+c^2+c|a_5|^2\right)(1+c)\left(1+c^2+2c|a_5|^2\right)\left(-\overline{ca_5}\left(1+\overline{c}|a_5|^2\right)-\overline{ca_5}(1+\overline{c})\right) \\ &\cdot\left(1+c|a_5|^2\right)-c^2a_5^2\left(1+c+c^2+c|a_5|^2\right)(1+c)\left(1+c^2+2c|a_5|^2\right)\left(1+\overline{c}|a_5|^2\right)\left(1+c|a_5|^2\right) \\ &+ca_5^2\left(1+c+c^2+c|a_5|^2\right)\left(1+c|a_5|^2\right)\left(1+c|a_5|^2\left(3+2c+c^2+c|a_5|^2\right)\right)\left(1+\overline{c}|a_5|^2\right) \\ &+ca_5^2(1+c)\left(1+c^2+2c|a_5|^2\right)\left(1+c|a_5|^2\right)\left(1+2c|a_5|^2+c^2|a_5|^2\right)\left(1+\overline{c}|a_5|^2\right)=0. \end{aligned} \tag{3.16}$$

For $a_5 = \frac{1}{2}$, solving the equations (3.15) and (3.16) we find $c = -0.625 + i0.780625$ Then we have B and M of the following forms:

$$B(z) = z\frac{(2z-1)\left(z-0.5+6.39697\times10^{-11}i\right)(z-0.3125+0.390312i)^2\left(z-2.51094\times10^{-11}-1.05107\times10^{-10}i\right)}{(2-z)\left(1-z\left(0.5+6.39697\times10^{-11}i\right)\right)(1-z(0.3125+0.390312i))^2\left(1-z\left(2.51094\times10^{-11}-1.05107\times10^{-10}i\right)\right)} \tag{3.17}$$

and

$$M(z) = (0.625-0.780625i)\,\frac{1-2z}{2-z}.$$

Then for the points $z \in \mathbb{D}$, we find

$$(B \circ M)(z) = B(z) \text{ and } B(z) = (B_2 \circ B_1)(z)$$

where

$$B_1(z) = z\frac{\left(z - 0.5 + 6.39697 \times 10^{-11} i\right)(z - 0.3125 + 0.390312i)}{(1 - z(0.5 + 6.39697 \times 10^{-11} i))(1 - z(0.3125 + 0.390312i))}$$

and

$$B_2(z) = z\frac{z - 2.44357 \times 10^{-11} - 1.15227 \times 10^{-11} i}{1 - z(2.44357 \times 10^{-11} - 1.15227 \times 10^{-11} i)}.$$

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