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On Nonlinear Fractional Neutral Differential Equation with the $\psi-$Caputo Fractional Derivative

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ABSTRACT: In this article, the solvability of fractional neutral differential equation involving $\psi-$Caputo fractional operator is considered using a Krasnoselskii's fixed point approach. Also, we establish the uniqueness of the solution under certain conditions. Ulam stabilities for the proposed problem are discussed. Finally, examples are displayed to aid the applicability of the theory results.

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1. Introduction

Fractional calculus is strong tool of mathematical analysis that studies derivatives and integrals of fractional order. Fractional differential equations (FDE's, for short) are used in many fields of engineering and sciences such as dynamical of biological systems [12], economy [33], theory of control [7], automatic systems [36], signal processing [11], hydro-mechanics and non-linear elasticity [14, 32].

Various real life problems can be modeled as differential equation. The study of existence of solution of these differential equation is interest object of mathematical analysis. The fixed point theorems are powerful technique to obtain the existence of solution of these problem. There are many of fixed point theorems can be applied to obtain the solution of mathematical models [24, 25]. Krasnoselskii's and Banach fixed point theorems play an important role to obtain the existence of solution of a lot of mathematical problems [35].

In 1940, Ulam purposed new role of the stability analysis of the solutions for functional equations [34]. In the next year, Hyer [15] considered another type of stability in the Banach space which was more generalized than the kind of Ulam stability and applied this stability approach to obtain the stability certain conditions of some functional equations. After that, Rassias [27] considered another approach of stability, this approach is more improved than Hyers stabitity. Rassias used this approach to study stability of FDE's [16, 28].

Recently, many research articles study the Ulam stabilities, see [21, 20, 13, 10, 8, 2, 22, 3, 19, 17, 18, 30]. In 2011, Ardjouni and Djoudi [6] studied the stability for neutral ordinary differential equations via fixed points. In 2019, Akbulut and Tunc [1], established the stability of solutions of neutral ordinary differential equations with multiple time delay. In the same year, Niazi [26], discussed Ulam stabilities for nonlinear fractional neutral differential equations in Caputo sense via Picard operator.

There are many definitions are used to define the fractional derivative such as Riemann-Liouville, Caputo, Erdélyi-Kober and Hadamard [23]. More recently, Almeida [4] considers new investigation of the fractional operator and called it $\psi-$Caputo derivative. This new approach is more generalized than Riemann-Liouville, Caputo, Erdélyi-Kober and Hadamard derivative operator approaches. After one year, Almeida et al.[5] investigated the uniqueness of solution of initial value problem (I.V.P, for short) of FDE in $\psi-$Caputo sense.

In this paper, we discuss the existence and uniqueness of the following FDE with delay

$$\begin{cases} {}^*D_{0^+}^{\alpha,\psi}[x(t) - H(t, x(t-\vartheta(t)))] = F(x(t), x(t-\vartheta(t))); \\ \alpha \in (0,1],\ t \in I = [0,1]; \\ \text{subject to I.V.} \\ x(t) = \sigma(t),\ t \in [\rho, 0]; \end{cases} \tag{1}$$

where ${}^*D^{\alpha,\psi}$ is $\psi-$Caputo derivative operator , the delay $\rho = \inf\{t - \vartheta(t) : t \in [0,1]\} \leq 0$, $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ and $\sigma : [\rho, 0] \to \mathbb{R}$.

2. Preliminaries

In this section, we consider some facts and basic results. We recall the following definition [3].

Definition 2.1. Let $C([\rho,1],\mathbb{R})$ be the vectorial space of all continuous functions $u : [\rho,1] \to \mathbb{R}$. Clearly, $C([\rho,1],\mathbb{R})$ is a complete normed space with the norm, $\|u\| = \max_{t\in[\rho,1]} |u(t)|$. Therefore, $C^n([\rho,1],\mathbb{R})$, $n \in \mathbb{N}$, be the vectorial space of all $n-$times continuous and differentiable functions from $[\rho,1]$ to $\mathbb{R}$.

Next, we recall the definitions of $\psi-$fractional integral and derivative operators [4, 5].

Definition 2.2. Let $I = [0,1]$ and $\psi \in C^n(I,\mathbb{R})$, be an increasing functions such that $\psi'(t) \neq 0$ for all $t \in I$. Consider an integrable function $u : I \to \mathbb{R}$. The

$\psi-$Riemann-Liouville fractional integral of order $\alpha > 0,\ \alpha \in \mathbb{R}$ of the function u is defined as

$$J_{0^+}^{\alpha,\psi} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(\zeta)(\psi(t) - \psi(\zeta))^{\alpha-1}\, u(\zeta)\, d\zeta\ ,$$

and the $\psi-$Riemann-Liouville fractional derivative of order $\alpha > 0,\ \alpha \in \mathbb{R}$ of the function u is defined as

$$D_{0^+}^{\alpha,\psi} u(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{1}{\psi'(t)} \frac{d}{dt})^n \int_0^t \psi'(\zeta)(\psi(t) - \psi(\zeta))^{n-\alpha-1}\, u(\zeta)\, d\zeta\ ,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integral part of α.

Definition 2.3. Let $\psi \in C^n(I,\mathbb{R})$, be an increasing function such that $\psi'(t) \neq 0$ for all $t \in I$. Consider an integrable function $u : I \to \mathbb{R}$. The $\psi-$Caputo fractional derivative of order $\alpha > 0,\ \alpha \in \mathbb{R}$ of the function u is defined as

$$^*D_{0^+}^{\alpha,\psi} u(t) = D_{0^+}^{\alpha,\psi}[u(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k],$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integral part of α and $u_\psi^{[k]}(t) = (\frac{1}{\psi'(t)} \frac{d}{dt})^k\, u(t)$.

We recall the following Lemma which was given in [5].

Lemma 2.4. *Suppose that $u : I \to \mathbb{R}$, then*
(1) If $u \in C(I,\mathbb{R})$, then $^*D_{0^+}^{\alpha,\psi}\ J_{0^+}^{\alpha,\psi} u(t) = u(t)$.
(2) If $u \in C^n(I,\mathbb{R})$, then

$$J_{0^+}^{\alpha,\psi}\ {^*D_{0^+}^{\alpha,\psi}}\ u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k.$$

Now we recall Krasnoselskii's fixed point theorem which was given in [31].

Theorem 2.5. ***(Krasnoselskii's fixed point theorem)*** *Let Υ be a Banach space. Suppose that Ω $(\Omega \neq \emptyset)$ be a convex, bounded and closed subset of Υ. Consider $\mathcal{T}_1 : \Upsilon \to \Upsilon$ and $\mathcal{T}_2 : \Omega \to \Upsilon$ are such that*
(1) $\mathcal{T}_1$ be a contraction.
(2) $\mathcal{T}_2$ is completely continuous.
(3) $x = \mathcal{T}_1 x + \mathcal{T}_2 y \Rightarrow x \in \Omega$ for all $y \in \Omega$.
Then, there exists $x^* \in \Omega$ such that $x^* = \mathcal{T}_1 x^* + \mathcal{T}_2 x^*$.

Now, we recall the definitions of these types of Ulam stability. For more details, see [29].

Definition 2.6. The Eq.(1) is said to be Ulam-Hyers stable (UHS for short) if, there exists $\lambda \in \mathbb{R}^+$ such that for every $\varepsilon > 0$ and each $u \in C([\rho, 1],\mathbb{R})$ solution of the inequality

$$|^*D_{0^+}^{\alpha,\psi}[u(t) - H(t, u(t - \vartheta(t)))] - F(u(t), u(t - \vartheta(t)))| \leq \varepsilon \quad , t \in I,$$

there exists a unique solution $x \in C([\rho,1],\mathbb{R})$ of Eq.(1) such that

$$|u(t)-x(t)| \le \lambda\, \varepsilon \quad ,\forall\, t \in [\rho,1].$$

Definition 2.7. The Eq.(1) is said to be generalized Ulam-Hyers stable (GUHS for short) if, there exists $\varphi \in C([\rho,1],\mathbb{R})$, $\varphi(0)=0$, such that for every $\varepsilon>0$ and each $u \in C([\rho,1],\mathbb{R})$ solution of the inequality

$$|{}^{*}D^{\alpha,\psi}_{0^+}[u(t)-H(t,u(t-\vartheta(t)))]-F(u(t),u(t-\vartheta(t)))| \le \varepsilon \quad ,\ t \in I,$$

there exists a unique solution $x \in C([\rho,1],\mathbb{R})$ of Eq.(1) such that

$$|u(t)-x(t)| \le \varphi(\varepsilon) \quad ,\forall\, t \in [\rho,1].$$

Definition 2.8. The Eq.(1) is called Ulam-Hyers-Rassias stable (UHRS for short) w.r.t $\varphi \in C([\rho,1],\mathbb{R})$, if there exists $\kappa_\varphi \in \mathbb{R}^+$ such that for every $\varepsilon>0$ and each $u \in C([\rho,1],\mathbb{R})$ solution of the inequality

$$|{}^{*}D^{\alpha,\psi}_{0^+}[u(t)-H(t,u(t-\vartheta(t)))]-F(u(t),u(t-\vartheta(t)))| \le \varepsilon\, \varphi(t) \quad ,t \in I\ , \tag{2}$$

there exists a unique solution $x \in C([\rho,1],\mathbb{R})$ of Eq.(1) such that

$$|u(t)-x(t)| \le \kappa_\varphi\, \varepsilon\, \varphi(t) \quad ,\forall\, t \in [\rho,1].$$

Definition 2.9. The Eq.(1) is said to be generalized Ulam-Hyers-Rassias stable (GUHRS for short) w.r.t $\varphi \in C([\rho,1],\mathbb{R})$, if there exists $\kappa_\varphi \in \mathbb{R}^+$ such that for each $u \in C([\rho,1],\mathbb{R})$ solution of the inequalities

$$|{}^{*}D^{\alpha,\psi}_{0^+}[u(t)-H(t,u(t-\vartheta(t)))]-F(u(t),u(t-\vartheta(t)))| \le \varphi(t) \quad ,\ t \in I\ ,$$

there exists a unique solution $u \in C([\rho,1],\mathbb{R})$ of the Eq.(1) such that

$$|u(t)-x(t)| \le \kappa_\varphi\, \varphi(t) \quad ,\forall\, t \in [\rho,t].$$

Let $H: I \times \mathbb{R} \to \mathbb{R}$ and $F: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. Then we study the Ulam stabilities of the following proposed problem

$$\begin{cases} {}^{*}D^{\alpha,\psi}_{0^+}[x(t)-H(t,x(t-\vartheta(t)))]=F(x(t),x(t-\vartheta(t))); \\ \alpha \in (0,1],\ t \in I=[0,1]; \\ \text{subject to initial value} \\ x(t)=\sigma(t),\ t \in [\rho,0]; \end{cases}$$

where ${}^{*}D^{\alpha,\psi}$ is $\psi-$Caputo derivative operator, $\rho=\inf\{t-\vartheta(t) : t \in [0,1]\} \le 0$, $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$, $\sigma : [\rho,0] \to \mathbb{R}$ are continues and $\psi \in C^1(I,\mathbb{R})$ be an increasing function such that $\psi'(t) \ne 0$ for all $t \in I$. Then, we have the following lemma [9].

Lemma 2.10. *The solution of Eq.*(1) *is equivalent to the following nonlinear integral equation*

$$\begin{aligned} x(t) &= \sigma(0)-H(0,\sigma(-\vartheta(0)))+H(t,x(t-\vartheta(t))) \\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\, F(x(s),x(s-\vartheta(s)))\, ds\ . \end{aligned}$$

3. Existence and Uniqueness

In this section we will obtain the existence of solution and uniqueness of the proposed neutral FDE (1). suppose that $r_0 \in \mathbb{R}^+$ and $\Omega = \{x \in C([\rho, 1], \mathbb{R}) : \|x\| \le r_0\}$. The Eq.(1) can be written as

$$(\mathcal{T}x)(t) = (\mathcal{T}_1 x)(t) + (\mathcal{T}_2 x)(t),$$

where

$$\mathcal{T}_1 : \Omega \to (CB([\rho, 1], \mathbb{R}) \quad , \qquad \mathcal{T}_2 : \Omega \to (CB([\rho, 1], \mathbb{R}),$$

such that

$$\begin{aligned}
&(\mathcal{T}_1 x)(t) = \sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, x(t - \vartheta(t))) \ , \\
&(\mathcal{T}_2 x)(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1} \ F(x(s), x(s - \vartheta(s))) \ ds,
\end{aligned}$$

where $t \in [\rho, 1]$ and $x \in C([\rho, 1], \mathbb{R})$.

We will study Eq.(1) under the following conditions:

(C1) the functions $H : I \times \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and there exist $p \in (0, 1), q \in \mathbb{R}^+$ such that

$$|H(t, x_1) - H(t, x_2)| < L|x_1 - x_2|,$$

$$|F(x_1, x_1) - F(y_1, y_2)| < K \sum_{i=1}^{2} |x_i - y_i|,$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, and $t \in [0, 1]$;

(C2) let $A^* =\mid F(0, 0) \mid$ and $B^* = \max\limits_{t \in I} \mid H(t, 0) \mid$ then

$$|\sigma(0) - H(0, \sigma(-\vartheta(0)))| + L \ r_0 + B^* + \frac{K \ r_0 + A^*}{\Gamma(\alpha + 1)} (\psi(1) - \psi(0))^{\alpha} \le r_0.$$

Theorem 3.1. *Let the conditions (C1) and (C2) hold. Then Eq.*(1) *has at leat one solution in* Ω.

Proof. The proof is done in the following 3 steps.

Step 1. $\mathcal{T}_1$ **is contraction.**

Let $x, y \in C([\rho, 1], \mathbb{R})$ are arbitrary and $t \in I$

$$|(\mathcal{T}_1 x)(t) - (\mathcal{T}_1 y)(t)| \le L|x(t) - y(t)|,$$

which implies that

$$\|\mathcal{T}_1 x - \mathcal{T}_1 y\| \le L\|x - y\|,$$

Thus, $\mathcal{T}_1$ is a contraction.

Step 2. $\mathcal{T}_2$ **is completely continuous.**

First, we will prove that $\mathcal{T}_2$ is continuous. Let $\{x_n\}$ be a sequence in $C([\rho, 1], \mathbb{R})$ such that $x_n \to x \in C([\rho, 1], \mathbb{R})$. Then, we get

$$\begin{aligned}|(\mathcal{T}_2x_n)(t)-(\mathcal{T}_2x)(t)| &\leq \tfrac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}|F(x_n(s),x_n(s-\vartheta(s)))\\ &\quad -F(x(s),x(s-\vartheta(s)))|\,ds\\ &\leq \tfrac{K}{\Gamma(\alpha+1)}(\psi(t)-\psi(0))^{\alpha}\|x_n-x\|\\ &\leq \tfrac{K}{\Gamma(\alpha+1)}(\psi(1)-\psi(0))^{\alpha}\|x_n-x\|.\end{aligned}$$

So, we have that

$$\|\mathcal{T}_2x_n-\mathcal{T}_2x\| \leq \tfrac{K}{\Gamma(\alpha+1)}(\psi(1)-\psi(0))^{\alpha}\|x_n-x\|.$$

Thus, $\|\mathcal{T}_2x_n-\mathcal{T}_2x\| \to 0$ as $n\to\infty$. Hence $\mathcal{T}_2$ is continuous operator. Therefore, for each $x,y\in C([\rho,1],\mathbb{R})$ and $t\in I$, we have

$$\begin{aligned}|F(x(t),y(t))| &\leq |F(x,y)-F(0,0)|+|F(0,0)|\\ &\leq K(\|x\|+\|y\|)+A^{*}.\end{aligned}$$

Therefore,

$$\begin{aligned}|\ (\mathcal{T}_2x)(t)\ | &\leq \tfrac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}|F(x(s),x(s,s-\vartheta(s)))|ds\\ &\leq \tfrac{2K+A^{*}}{\Gamma(\alpha+1)}(\psi(t)-\psi(0))^{\alpha},\end{aligned}$$

for all $t\in I$. Hence we have

$$\|\ \mathcal{T}_2x\ \| \leq \tfrac{2K+A^{*}}{\Gamma(\alpha+1)}(\psi(1)-\psi(0))^{\alpha}.$$

Thus $\mathcal{T}_2$ is bounded. Furthermore, if we choose $t_1,t_2\in I$ such that $t_1<t_2$, then we get

$$\begin{aligned}&|(\mathcal{T}_2x)(t_2)-(\mathcal{T}_2x)(t_1)|\\ &=|\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi^{'}(s)(\psi(t_2)-\psi(s))^{\alpha-1}F(x(s),x(s-\vartheta(s)))\,ds\\ &\quad -\tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}F(x(s),x(s-\vartheta(s)))\,ds|\\ &\leq |\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi^{'}(s)(\psi(t_2)-\psi(s))^{\alpha-1}F(x(s),x(s-\vartheta(s)))\,ds\\ &\quad -\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}F(x(s),x(s-\vartheta(s)))\,ds|\\ &\quad +|\tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}F(x(s),x(s-\vartheta(s)))\,ds\\ &\quad -\tfrac{1}{\Gamma(\alpha)}\int_0^{t_1}\psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}F(x(s),x(s-\vartheta(s)))\,ds|\\ &\leq \tfrac{1}{\Gamma(\alpha)}\int_0^{t_2}\psi^{'}(s)[(\psi(t_2)-\psi(s))^{\alpha-1}-(\psi(t_1)-\psi(s))^{\alpha-1}]\ |\ F(x(s),x(s-\vartheta(s)))\ |\ \ ds\\ &\quad +\tfrac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi^{'}(s)(\psi(t_1)-\psi(s))^{\alpha-1}\ |\ F(x(s),x(s-\vartheta(s)))\ |\ \ ds\\ &\leq \tfrac{2K\ r_0+A^{*}}{\Gamma(\alpha+1)}[(\psi(t_2)-\psi(0))^{\alpha}-(\psi(t_1)-\psi(0))^{\alpha}].\end{aligned}$$

Since ψ is continuous, then we have that $|(\mathcal{T}_2x)(t_2)-(\mathcal{T}_2x)(t_1)|\to 0$ as $t_1\to t_2$. Thus $\mathcal{T}_2(\Omega)$ is relatively compact. From Arzela-Ascoli-theorem, we obtain $\mathcal{T}_2$ is compact. Hence $\mathcal{T}_2$ is completely continuous.

Step 3. Finding the fixed poind of $\mathcal{T}$.
Let $x, y \in \Omega$. We get

$$\begin{aligned}
&|(\mathcal{T}_1 x)(t) + (\mathcal{T}_2 y)(t)| \\
&= |\sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, x(t - \vartheta(t))) \\
&\quad + \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s)))\, ds| \\
&\leq |\sigma(0) - H(0, \sigma(-\vartheta(0)))| + |H(t, x(t - \vartheta(t)))| + \\
&\quad + |\tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1} F(x(s), x(s - \vartheta(s)))\, ds| \\
&\leq |\sigma(0) - H(0, \sigma(-\vartheta(0)))| + L\, r_0 + B^* + \tfrac{K\, r_0 + A^*}{\Gamma(\alpha+1)} (\psi(t) - \psi(0))^{\alpha} \\
&\leq |\sigma(0) - H(0, \sigma(-\vartheta(0)))| + L\, r_0 + B^* + \tfrac{K\, r_0 + A^*}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^{\alpha} \\
&\leq r_0
\end{aligned}$$

Thus, the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy all conditions of Theorem 2.5. Hence there exists $x^* \in \Omega$ such that x^* is solution of Eq.(1). □

Theorem 3.2. *Suppose that the conditions (C1) and (C2) hold. Let,*
(C3) $L + \frac{2K}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^{\alpha} < 1.$
Then the Eq.(1) *has unique solution.*

Proof. We apply Banach contraction theorem to prove $\mathcal{T}$ has a unique fixed point. Let $x, y \in C([\rho, 1], \mathbb{R})$. Then, we have

$$\begin{aligned}
|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| &\leq L|x(t) - y(t)| + \tfrac{2K\|x-y\|}{\Gamma(\alpha)} \textstyle\int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\, ds \\
&\leq L + \tfrac{2K}{\Gamma(\alpha+1)} (\psi(t) - \psi(0))^{\alpha} \\
&\leq L + \tfrac{2K}{\Gamma(\alpha+1)} (\psi(1) - \psi(0))^{\alpha} \\
&\leq 1.
\end{aligned}$$

Thus Eq.(1) has unique solution. □

4. Ulam Stabilities

In this part, various Ulam stability types will be considered.

Lemma 4.1. *Let* $\alpha \in (0, 1)$, *if* $z \in C([\rho, 1], \mathbb{R})$ *is the solution of the inequality of definition 2.6, then* z *is the solution of the following inequality*

$$|z(t) - N(t)| \leq \left(\tfrac{(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)\varepsilon,$$

where

$$\begin{aligned}
N(t) &= \sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, z(t - \vartheta(t))) \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1} F(z(s), z(s - \vartheta(s)))\, ds.
\end{aligned}$$

Proof. Let $z \in C([\rho,1],\mathbb{R})$ be any solution of the inequality of definition 2.6, then there exists $\Theta \in C([\rho,1],\mathbb{R})$ dependent on z such that

$$\begin{cases} {}^{*}D_{0^+}^{\alpha,\psi}[z(t) - H(t, z(t-\vartheta(t)))] = F(z(t), z(t-\vartheta(t))) + \Theta(t)\ ; \\ \alpha \in (0,1], t \in I = [0,1]\ ; \\ \text{subject to initial value} \\ z(t) = \sigma(t),\ t \in [\rho, 0]\ ; \end{cases} \tag{3}$$

and

$$|\Theta(t)| \leq \varepsilon \quad , \forall t \in I.$$

Thus, Eq.(3) is equivalent to the following equation

$$\begin{aligned} z(t) = {} & \sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, z(t-\vartheta(t))) \\ & + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\ F(z(s), z(s-\vartheta(s)))\ ds \\ & + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{(\alpha-1)}\ \Theta(s)\ ds. \end{aligned}$$

Let

$$\begin{aligned} N(t) = {} & \sigma(0) - H(0, \sigma(-\vartheta(0))) + H(t, z(t-\vartheta(t))) \\ & + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\ F(z(s), z(s-\vartheta(s)))\ ds. \end{aligned}$$

Thus, we have

$$|z(t) - N(t)| \leq \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{\alpha-1}\ |\Theta(s)|\ ds \leq \tfrac{1}{\Gamma(\alpha+1)}\ (\psi(1) - \psi(0))^{\alpha}\ \varepsilon.$$

□

Theorem 4.2. *Suppose that (C1)-(C3) hold. Then the Eq.(1) is UHS and consequently GUHS.*

Proof. Let $z \in C([\rho,1],\mathbb{R})$ be a solution of the inequality of definition 2.6 and x be the unique solution of Eq.(1), then we get $|N(t)| \leq \varepsilon$ for all $t \in I$ and

$$|z(t) - x(t)| \leq |z(t) - N(t)| + |N(t) - x(t)|.$$

From Lemma 4.1, we get

$$\begin{aligned} |z(t) - x(t)| \leq {} & (\tfrac{(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})\varepsilon_1 + L|z(t) - x(t)| \\ & + \tfrac{1}{\Gamma(\alpha)} \int_0^t \psi^{'}(s)(\psi(t) - \psi(s))^{(\alpha-1)}\ 2K\ |z(s) - x(s)|\ ds \\ \leq {} & (\tfrac{(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})\varepsilon_1 + L|z(t) - x(t)| + \tfrac{2K}{\Gamma(\alpha+1)}(\psi(1) - \psi(0))^{\alpha}\ |z(t) - x(t)|\ , \end{aligned}$$

therefore, we get

$$\|z - x\| \leq (\tfrac{(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)})\varepsilon_1 + \mathbb{L}\|z - x\|,$$

where

$$\mathbb{L} = 1 - [L + \tfrac{2K}{\Gamma(\alpha+1)}(\psi(1) - \psi(0))^{\alpha}].$$

Then , we get

$$\|z-x\| \leq \lambda\, \varepsilon,$$

where

$$\lambda = \frac{\left(\frac{(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)}{1-\mathbb{L}}.$$

Thus the Eq.(1) is UHS. Therefore, if we put $\varphi(\varepsilon) = \lambda\, \varepsilon$, then we get that $\varphi(0) = 0$ and

$$\|z-x\| \leq \varphi(\varepsilon).$$

Then, the Eq.(1) is GUHS. □

Lemma 4.3. *Suppose that the following condition holds:*

(C4) If $\phi \in C([\rho,1],\mathbb{R})$ is increasing, then there exists $\mu_\phi \in \mathbb{R}^+$ such that for every $t \in I$, the following inequality hold

$$^{*}J_{0^+}^{\alpha,\psi}\phi(t) \leq \mu_\phi\, \phi(t)\ .$$

If $z \in C([\rho,1],\mathbb{R})$ is the solution of the inequality (2)*, then z is the solution of the following inequality*

$$|z(t)-N(t)| \leq \mu_\phi\left(\frac{(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)\phi(t)\, \varepsilon.$$

Proof. From Lemma 4.1, we get

$$|z(t)-N(t)| \leq \frac{1}{\Gamma(\alpha)}\int_0^t \psi^{'}(s)(\psi(t)-\psi(s))^{\alpha-1}\, |\Theta(s)|\, ds.$$

From (C4), we have that

$$|z(t)-N(t)| \leq \mu_\phi\left(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)\phi(t)\, \varepsilon.$$

□

Theorem 4.4. *Consider the Conditions (C1)-(C4) hold. Then the Eq.*(1) *is UHRS and GUHRS .*

Proof. Let $z \in C([\rho,1],\mathbb{R})$ be solution of the inequality (2) and x be the unique solution of Eq.(1). From Lemma 4.3, we get

$$\|z-x\| \leq \mu_\phi\left(\frac{(\psi(1)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\right)\phi_1(t)\, \varepsilon + \mathbb{L}\, \|z-x\|.$$

So, we have that

$$\|z-x\| \leq \mu_\phi\, \lambda\, \phi(t)\varepsilon.$$

Thus the Eq.(1) is UHRS. Therefore, if we put $\varepsilon = 1$, then the Eq.(1) is GUHRS.

□

5. Applications

The following examples are applications to the previous theoretical results.

Example 5.1. Consider the following $\psi-$Caputo FDE

$$\begin{cases} {}^*D_{0^+}^{\frac{1}{3},\psi}[x(t) - \frac{te^{-t}}{10}x(t-0.1)] = \frac{1}{10}\tan^{-1}(x(t)) + \frac{|x(t-0.1)|}{14+|x(t-0.1)|}; \\ t \in I = [0,1], \\ \text{subject to the nonlocal conditions} \\ x(t) = 0.2 \quad , t \in [-0.1, 0] \end{cases} \tag{4}$$

where $\psi(t) = \sqrt{1+t}$, for all $t \in [0,1]$. Clearly, ψ is increasing on $[0,1]$ and $\psi \in C^1([0,1],\mathbb{R})$. Therefore,

$$H(t,x) = \frac{te^{-t}}{10}x,$$

also

$$F(t,x,y) = \frac{1}{10}\tan^{-1}(x) + \frac{|y|}{14+|y|}.$$

It is clear that, H, F are continuous. Since,

$$|H(t,x_1) - H(t,x_2)| \leq \frac{1}{10}|x_1 - x_2|,$$

$$|F(t,x_1,y_1) - F(t,x_2,y_2)| \leq \frac{1}{10}(|x_1 - x_2| + |y_1 - y_2|),$$

for all $x, y, x_1, y_1, x_2, y_2 \in \mathbb{R}$ and $t \in [0,1]$. Thus, the condition (C1) holds with

$$L = K = \frac{1}{10} \ ,$$

therefore

$$A^* = 0 \quad , \quad B^* = 0 \quad , \quad \sigma(0) = 0.2.$$

The inequality of (C2)

$$|\sigma(0) - H(0, \sigma(-\vartheta(0)))| + L\, r_0 + B^* + \frac{K\, r_0 + A^*}{\Gamma(\alpha+1)}(\psi(1) - \psi(0))^\alpha \leq r_0,$$

has the following form

$$0.2 + \frac{r_0}{10} + \frac{r_0(\sqrt{2}-1)^{\frac{1}{3}}}{10\Gamma(\frac{4}{3})} \leq r_0.$$

Hence (C2) is hold and $r_0 \geq 0.2447531$. Similarly, we get: $L + \frac{2K}{\Gamma(\alpha+1)}(\psi(1)-\psi(0))^\alpha = 0.11657002573 < 1$. Hence the condition (C3) holds. So, it is implies that, the Eq.(4) has a unique solution. Hence, the Eq.(4) is UHS, GUHS, UHRS and GUHRS.

Example 5.2. Consider the following $\psi-$Caputo FDE

$$\begin{cases} {}^{*}D_{0^+}^{\frac{1}{2},\psi}[x(t) - \frac{t}{9}\sin(x(t-0.1))] = \frac{1}{12}x(t) + \frac{1}{10}x(t-0.1) \\ t \in I = [0,1], \\ \text{subject to the nonlocal conditions} \\ x(t) = 0.2 \quad , t \in [-0.1, 0] \end{cases} \tag{5}$$

where $\psi(t) = \frac{t^2+t}{2}$, for all $t \in [0,1]$. Clearly, ψ is increasing on $[0,1]$ and $\psi \in C^1([0,1],\mathbb{R})$. Therefore,

$$H(t,x) = \frac{t}{9}\sin(x),$$

also

$$F(t,x,y) = \frac{1}{12}x + \frac{1}{10}y.$$

It is clear that, H, F are continuous. Since,

$$|H(t,x_1) - H(t,x_2)| \leq \frac{1}{9}|x_1 - x_2|,$$

$$|F(t,x_1,y_1) - F(t,x_2,y_2)| \leq \frac{1}{10}(|x_1 - x_2| + |y_1 - y_2|),$$

for all $x, y, x_1, y_1, x_2, y_2 \in \mathbb{R}$ and $t \in [0,1]$. Thus, the conditions (C1) holds with

$$L = \frac{1}{9} \quad , K = \frac{1}{10}$$

therefore

$$A^* = 0 \quad , \quad B^* = 0 \quad , \quad \sigma(0) = 0.2.$$

The inequality of (C2)

$$|\sigma(0) - H(0, \sigma(-\vartheta(0)))| + L\, r_0 + B^* + \frac{K\, r_0 + A^*}{\Gamma(\alpha+1)}(\psi(1) - \psi(0))^\alpha \leq r_0,$$

has the following form

$$0.2 + \frac{r_0}{10} + \frac{r_0}{10\Gamma(\frac{3}{2})} \leq r_0.$$

Hence (C2) is hold and $r_0 \geq 0.25390377047$. Similarly, we get: $L + \frac{2K}{\Gamma(\alpha+1)}(\psi(1) - \psi(0))^\alpha = 0.32471910112 < 1$. Hence the condition (C3) holds . So, it is implies that, the Eq.(5) has a unique solution. Hence, the Eq.(5) is UHS, GUHS, UHRS and GUHRS.

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