Journal of
Mathematics
and Applications

JMA No 45, pp 5-20 (2022)

On the Hybrid Caputo-Proportional Fractional Differential Inclusions in Banach Spaces

Mohamed I. Abbas

Abstract: The current article concerns an existence criteria of solutions of nonlinear fractional differential inclusions in the sense of the hybrid Caputo-proportional fractional derivatives in Banach space. The investigation of the main result relies on the set-valued issue of Mönch fixed point theorem incorporated with the Kuratowski measure of noncompactness.

AMS Subject Classification: 34A08, 34A12, 34A08, 34A12, 34A37, 34G20.
Keywords and Phrases: Hybrid Caputo-proportional fractional derivatives; Measure of non-compactness; Mönch fixed point theorem.

1. Introduction

It is recently seen that there is a wide-spread of fractional differential systems because of their great relevance to reality and their dignified influence in describing several real-world problems in physics, mechanics and engineering. For intance, we refer the reader to the monographs of Baleanu et al.[7], Hilfer [21], Kilbas et al. [24], Mainardi [26], Miller and Ross [27], Podlubny [30], Samko et al. [32] and the papers [17, 33].

Due to the importance of fractional differential inclusions in mathematical modeling of problems in game theory, stability, optimal control, and so on. For this reason, many contributions have been investigated by some researchers [1, 4, 11, 12, 13, 18, 29].

On the other hand, the theory of measure of non-compactness is an essential tool in investigating the existence of solutions for nonlinear integral and differential equations, see, for example, the recent papers [5, 10, 15, 19, 31] and the references existing therein.

In [14], Benchohra et al. studied the existence of solutions for the fractional differential inclusions with boundary conditions

$$\begin{cases} {}^{C}\mathcal{D}^{r}y(t) \in G\left(t, y(t)\right), & \text{a.e. on } [0,T], \quad 1 < r < 2, \\ y(0) = y_0, \quad y(T) = y_T, \end{cases}$$

where ${}^{C}\mathcal{D}^{r}$ is the Caputo fractional derivative, $G : [0,T] \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is a multi-valued map, $y_0, y_T \in \mathbf{E}$ and $(\mathbf{E}, |\cdot|)$ is a Banach space.

Motivated by the above work, in this paper, we will extend the Caputo fractional derivative with a broader and more general one, which can be written as a Riemann-Liouville integral of a proportional derivative, or in some important special cases as a linear combination of a Riemann-Liouville integral and a Caputo derivative. To be more precise we will study the existence of solutions for the following nonlinear fractional differential inclusions with the hybrid Caputo-proportional fractional derivatives

$$\begin{cases} {}_{0}^{PC}\mathcal{D}_{t}^{\alpha}x(t) \in F\left(t, x(t)\right), & \text{a.e. on } \mathbf{J} := [0,b], \quad 0 < \alpha < 1, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where ${}_{0}^{PC}\mathcal{D}_{t}^{\alpha}$ denotes the hybrid proportional-Caputo fractional derivative of order α, $(\mathbf{E}, |\cdot|)$ is a Banach space, $\mathfrak{P}(\mathbf{E})$ is the family of all nonempty subsets of $\mathbf{E}$, $x_0 \in \mathbf{E}$ and $F : \mathbf{J} \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is a given multi-valued map. We study the inclusion problem (1.1) in the case where the right hand side is convex-valued by means of the set-valued issue of Mönch fixed point theorem incorporated with the Kuratowski measure of non-compactness.

It is worth noting that the relevant results of fractional differential inclusions with the hybrid Caputo-proportional fractional derivatives are scarce. So the main goal of the present work is to contribute to the development of this area. Further, the topic of research has attracted lots of interests as a powerful tool for modeling scientific phenomena. Therefore, we refer the reader to some recent results which can be helpful for more related extensions or generalizations of the results in this paper in the future research works, see [22, 23, 28].

2. Preliminaries

First, we recall from [6] the following definition of the proportional (conformable) derivative of order α:

$${}_{0}^{P}D^{\alpha}g(t) = k_1(\alpha, t)g(t) + k_0(\alpha, t)g'(t),$$

where g is differentiable function and $k_0, k_1 : [0,1] \times \mathbb{R} \to [0,\infty)$ are continuous functions of the variable t and the parameter $\alpha \in [0,1]$ which satisfy the following conditions for all $t \in \mathbb{R}$:

$$\lim_{\alpha \to 0^+} k_0(\alpha,t) = 0, \quad \lim_{\alpha \to 1^-} k_0(\alpha,t) = 1, \quad k_0(\alpha,t) \neq 0, \ \alpha \in (0,1], \tag{2.1}$$

$$\lim_{\alpha \to 0^+} k_1(\alpha,t) = 1, \quad \lim_{\alpha \to 1^-} k_1(\alpha,t) = 0, \quad k_1(\alpha,t) \neq 0, \ \alpha \in [0,1). \tag{2.2}$$

Next, we explore the new definitions of the generalized hybrid proportional-Caputo fractional derivative.

Definition 2.1. [8] The hybrid Caputo-proportional fractional derivative of order $\alpha \in (0,1)$ of a differentiable function $g(t)$ is given by

$${}_0^{PC}\mathcal{D}_t^{\alpha} g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \big(k_1(\alpha,\tau) g(t) + k_0(\alpha,\tau) g'(t)\big)(t-\tau)^{-\alpha} \, d\tau, \tag{2.3}$$

where the function space domain is given by requiring that g is differentiable and both g and g' are locally L^1 functions on the positive reals.

Definition 2.2. [8] The inverse operator of the hybrid Caputo-proportional fractional derivative of order is given by

$${}_0^{PC}\mathcal{I}_t^{\alpha} g(t) = \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{{}_0^{RL}\mathcal{D}_u^{1-\alpha} g(u)}{k_0(\alpha,u)} \, du, \tag{2.4}$$

where ${}_0^{RL}\mathcal{D}_u^{1-\alpha}$ denotes the Riemann-Liouville fractional derivative of order $1-\alpha$ and is given by

$${}_0^{RL}\mathcal{D}_u^{1-\alpha} g(u) = \frac{1}{\Gamma(\alpha)} \frac{d}{du} \int_0^u (u-s)^{\alpha-1} g(s) \, ds. \tag{2.5}$$

For more details, we refer the reader to the book of Kilbas et al. [24].

Proposition 2.3. *[8] The following inversion relations:*

$${}_0^{PC}\mathcal{D}_t^{\alpha} \; {}_0^{PC}\mathcal{I}_t^{\alpha} g(t) = g(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t \to 0} \; {}_0^{RL}\mathcal{I}_t^{\alpha} g(t), \tag{2.6}$$

$${}_0^{PC}\mathcal{I}_t^{\alpha} \; {}_0^{PC}\mathcal{D}_t^{\alpha} g(t) = g(t) - \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) g(0) \tag{2.7}$$

are satisfied.

Proposition 2.4. *[8] The hybrid Caputo-proportional fractional derivative operator* ${}_0^{PC}\mathcal{D}_t^{\alpha}$ *is non-local and singular.*

Remark 2.5. [8] In the limiting cases $\alpha \to 0$ and $\alpha \to 1$, we recover the following special cases:

$$\begin{aligned}\lim_{\alpha\to 0} \ {}^{PC}_{\ 0}\mathcal{D}^{\alpha}_{t} g(t) &= \int_0^t g(\tau)\ d\tau,\\ \lim_{\alpha\to 1} \ {}^{PC}_{\ 0}\mathcal{D}^{\alpha}_{t} g(t) &= g(t).\end{aligned}$$

Denote by $C(\mathbf{J},\mathbf{E})$ the Banach space of all continuous functions from $\mathbf{J}$ to $\mathbf{E}$ with the norm $\|x\| = \sup_{t\in\mathbf{J}} |x(t)|$. By $L^1(\mathbf{J},\mathbf{E})$, we indicate the space of Bochner integrable functions from $\mathbf{J}$ to $\mathbf{E}$ with the norm $\|x\|_1 = \int_0^b |x(t)|\, dt$.

2.1. Multi-valued maps analysis

Let the Banach space be $(\mathbf{E}, |\cdot|)$. The expressions we have used are $\mathfrak{P}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \neq \emptyset\}$, $\mathfrak{P}_{\mathbf{cl}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is closed}\}$, $\mathfrak{P}_{\mathbf{bd}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is bounded}\}$, $\mathfrak{P}_{\mathbf{cp}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is compact}\}$, $\mathfrak{P}_{\mathbf{cvx}}(\mathbf{E}) = \{Z \in \mathfrak{P}(\mathbf{E}) : Z \textit{ is convex}\}$.

- A multi-valued map $\mathfrak{U} : \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is convex (closed) valued, if $\mathfrak{U}(x)$ is convex (closed) for all $x \in \mathbf{E}$.
- $\mathfrak{U}$ is bounded on bounded sets if $\mathfrak{U}(B) = \cup_{x\in B}\mathfrak{U}(x)$ is bounded in $\mathbf{E}$ for any $B \in \mathfrak{P}_{\mathbf{bd}}(\mathbf{E})$, i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in \mathfrak{U}(x)\}\} < \infty$.
- $\mathfrak{U}$ is called upper semi-continuous on $\mathbf{E}$ if for each $x^* \in \mathbf{E}$, the set $\mathfrak{U}(x^*)$ is nonempty, closed subset of $\mathbf{E}$, and if for each open set N of $\mathbf{E}$ containing $\mathfrak{U}(x^*)$, there exists an open neighborhood N^* of x^* such that $\mathfrak{U}(N^*) \subset N$.
- $\mathfrak{U}$ is completely continuous if $\mathfrak{U}(B)$ is relatively compact for each $B \in \mathfrak{P}_{\mathbf{bd}}(\mathbf{E})$.
- If $\mathfrak{U}$ is a multi-valued map that is completely continuous with nonempty compact values, then $\mathfrak{U}$ is u.s.c. if and only if $\mathfrak{U}$ has a closed graph (that is, if $x_n \to x_0, y_n \to y_0$, and $y_n \in \mathfrak{U}(x_n)$, then $y_0 \in \mathfrak{U}(x_0)$).

For more details about multi-valued maps, we refer to the book of Deimling [16].

Definition 2.6. A multi-valued map $F : \mathbf{J} \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $u \in \mathbf{E}$;

(ii) $x \mapsto F(t,x)$ is upper semi-continuous for almost all $t \in \mathbf{J}$.

We define the set of the selections of a multi-valued map F by

$$\mathcal{S}_{F,x} := \{f \in L^1(\mathbf{J},\mathbf{E}) : f(t) \in F(t,x(t)) \text{ for a.e. } t \in \mathbf{J}\}.$$

Lemma 2.7. *[25] Let $\mathbf{J}$ be a compact real interval and $\mathbf{E}$ be a Banach space. Let F be a multi-valued map satisfying the Caratheodory conditions with the set of L^1-selections $\mathcal{S}_{F,u}$ nonempty, and let $\Theta : L^1(\mathbf{J}, \mathbf{E}) \to C(\mathbf{J}, \mathbf{E})$ be a linear continuous mapping. Then the operator*

$$\Theta \circ \mathcal{S}_{F,x} : C(\mathbf{J}, \mathbf{E}) \to \mathfrak{P}_{\mathbf{bd,cl,cvx}}(C(\mathbf{J}, \mathbf{E})), \quad x \mapsto (\Theta \circ \mathcal{S}_{F,x})(x) := \Theta(\mathcal{S}_{F,x})$$

is a closed graph operator in $C(\mathbf{J}, \mathbf{E}) \times C(\mathbf{J}, \mathbf{E})$.

2.2. Measure of non-compactness

We specify this part of the paper to explore some important details of the Kuratowski measure of non-compactness.

Definition 2.8. [9] Let $\Lambda_{\mathbf{E}}$ be the family of bounded subsets of a Banach space $\mathbf{E}$. We define the Kuratowski measure of non-compactness $\kappa : \Lambda_{\mathbf{E}} \to [0, \infty]$ of $\mathbf{B} \in \Lambda_{\mathbf{E}}$ as

$$\kappa(\mathbf{B}) = \inf\{\epsilon > 0 : \mathbf{B} \subset \bigcup_{j=1}^{m} \mathbf{B}_j \text{ and } \operatorname{diam}(\mathbf{B}_j) \leq \epsilon\}.$$

Lemma 2.9. *[9] Let $\mathbf{C}, \mathbf{D} \subset \mathbf{E}$ be bounded, the Kuratowski measure of non-compactness possesses the next characteristics:*

i. $\kappa(\mathbf{C}) = 0 \Leftrightarrow \mathbf{C}$ *is relatively compact;*

ii. $\mathbf{C} \subset \mathbf{D} \Rightarrow \kappa(\mathbf{C}) \leq \kappa(\mathbf{D})$*;*

iii. $\kappa(\mathbf{C}) = \kappa(\overline{\mathbf{C}})$*, where $\overline{\mathbf{C}}$ is the closure of $\mathbf{C}$;*

iv. $\kappa(\mathbf{C}) = \kappa(conv(\mathbf{C}))$*, where $conv(\mathbf{C})$ is the convex hull of $\mathbf{C}$;*

v. $\kappa(\mathbf{C} + \mathbf{D}) \leq \kappa(\mathbf{C}) + \kappa(\mathbf{D})$*, where $\mathbf{C} + \mathbf{D} = \{u + v : u \in \mathbf{C},\ v \in \mathbf{D}\}$;*

vi. $\kappa(\nu\mathbf{C}) = |\nu|\kappa(\mathbf{C})$*, for any $\nu \in \mathbb{R}$.*

Theorem 2.10. *(Mönch's fixed point theorem) Let Ω be a closed and convex subset of a Banach space $\mathbf{E}$; $\mathcal{U}$ a relatively open subset of Ω, and $\mathcal{N} : \overline{\mathcal{U}} \to \mathfrak{P}(\Omega)$. Assume that graph $\mathcal{N}$ is closed, $\mathcal{N}$ maps compact sets into relatively compact sets and for some $x_0 \in \mathcal{U}$, the following two conditions are satisfied:*

(i) *$G \subset \overline{\mathcal{U}}$, $G \subset conv(x_0 \cup \mathcal{N}(G))$, $\overline{G} = \overline{C}$ implies $\overline{G}$ is compact, where C is a countable subset of G;*

(ii) $x \notin (1 - \mu)x_0 + \mu\mathcal{N}(x) \quad \forall u \in \overline{\mathcal{U}} \backslash \mathcal{U}, \quad \mu \in (0, 1)$.

Then there exists $x \in \overline{\mathcal{U}}$ with $x \in \mathcal{N}(x)$.

Theorem 2.11. *[20] Let $\mathbf{E}$ be a Banach space and $C \subset L^1(\mathbf{J}, \mathbf{E})$ countable with $|u(t)| \leq h(t)$ for a.e. $t \in \mathbf{J}$, and every $u \in C$; where $h \in L^1(\mathbf{J}, \mathbb{R}_+)$. Then the function $z(t) = \kappa(C(t))$ belongs to $L^1(\mathbf{J}, \mathbb{R}_+)$ and satisfies*

$$\kappa\Big(\Big\{ \int_0^b u(\tau)\, d\tau : u \in C\Big\}\Big) \leq 2 \int_0^b \kappa(C(\tau))\, d\tau.$$

3. Main results

We start this section with the definition of a solution of the inclusion problem (1.1).

Definition 3.1. A function $x \in C(\mathbf{J}, \mathbf{E})$ is said to be a solution of the inclusion problem (1.1) if there exist a function $f \in L^1(\mathbf{J}, \mathbf{E})$ with $f(t) \in F(t, x(t))$ for a.e. $t \in \mathbf{J}$, such that ${}^{PC}_{0}D^{\alpha}_{t}x(t) = f(t)$ on $\mathbf{J}$, and the condition $x(0) = x_0$ is satisfied.

Lemma 3.2. *For $0 < \alpha \leq 1$ and $h \in C(\mathbf{J}, \mathbb{R})$ the solution x of the linear hybrid Caputo-proportional fractional differential equation*

$$\begin{cases} {}^{PC}_{0}D^{\alpha}_{t}x(t) = h(t), \quad t \in \mathbf{J}, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

is given by the following integral equation

$$\begin{aligned} x(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) x_0 \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} h(\tau)\, d\tau\, du, \quad t \in \mathbf{J}. \end{aligned} \tag{3.2}$$

Proof. Applying the operator ${}^{PC}_{0}I^{\alpha}_{t}(\cdot)$ on both sides of (3.1), we get

$${}^{PC}_{0}I^{\alpha}_{t}\, {}^{PC}_{0}D^{\alpha}_{t}x(t) = {}^{PC}_{0}I^{\alpha}_{t}h(t).$$

Using (2.4) and (2.5) together with Proposition 2.3, we get

$$\begin{aligned} x(t) - \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds\right) x(0) &= \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds\right) \frac{{}^{RL}_{0}D^{1-\alpha}_{u}h(u)}{k_0(\alpha, u)}\, du \\ = \frac{1}{\Gamma(\alpha)} \int_0^t \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds\right) &\frac{1}{k_0(\alpha, u)} \frac{d}{du} \int_0^u (u-\tau)^{\alpha-1} h(\tau)\, d\tau\, du \end{aligned} \tag{3.3}$$

Using the following Leibniz's rule:

$$\frac{d}{du} \int_{a_1(u)}^{a_2(u)} w(u, \tau)\, d\tau = \int_{a_1(u)}^{a_2(u)} \frac{\partial}{\partial u} w(u, \tau)\, d\tau + w(u, a_2(u)) a_2'(u) - w(u, a_1(u)) a_1'(u),$$

where $w(u, \tau) = (u-\tau)^{\alpha-1} h(\tau)$, $a_1(u) = 0$, and $a_2(u) = u$, we obtain that

$$\frac{d}{du} \int_0^u (u-\tau)^{\alpha-1} h(\tau)\, d\tau = (\alpha - 1) \int_0^u (u-\tau)^{\alpha-2} h(\tau)\, d\tau. \tag{3.4}$$

Therefore, the substitution from (3.4) in (3.3), we get

$$\begin{aligned} x(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) x_0 \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} h(\tau)\, d\tau\, du. \end{aligned}$$

This completes the proof. □

Remark 3.3. The result of Lemma 3.2 is true not only for real valued functions $x \in C(\mathbf{J}, \mathbb{R})$ but also for a Banach space functions $x \in C(\mathbf{J}, \mathbf{E})$.

Lemma 3.4. *Assume that $F : \mathbf{J} \times \mathbf{E} \to \mathfrak{P}(\mathbf{E})$ satisfies Carathèodory conditions, i.e., $t \mapsto F(t, x)$ is measurable for every $x \in \mathbf{E}$ and $x \mapsto F(t, x)$ is continuous for every $t \in \mathbf{J}$. A function $x \in C(\mathbf{J}, \mathbf{E})$ is a solution of the inclusion problem (1.1) if and only if it satisfies the integral equation*

$$\begin{aligned} x(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) x_0 \\ &\quad + \frac{1}{\Gamma(\alpha - 1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds\right) \frac{(u - \tau)^{\alpha - 2}}{k_0(\alpha, u)} f(\tau)\, d\tau\, du, \end{aligned} \tag{3.5}$$

where $f \in L^1(\mathbf{J}, \mathbf{E})$ with $f(t) \in F(t, x(t))$ for a.e. $t \in \mathbf{J}$.

Now, we are ready to present the main result of the current paper.

Theorem 3.5. *Let $\varrho > 0$, $\mathcal{K} = \{x \in \mathbf{E} : \|x\| \leq \varrho\}$, $\mathcal{U} = \{x \in C(\mathbf{J}, \mathbf{E}) : \|x\| < \varrho\}$, and suppose that:*

(H1) *The multi-valued map $F : \mathbf{J} \times \mathbf{E} \to \mathfrak{P}_{\mathbf{cp,cvx}}(\mathbf{E})$ is Carathèodory,*

(H2) *For each $\varrho > 0$, there exists a function $\varphi \in L^1(\mathbf{J}, \mathbb{R}_+)$ such that*

$$\|F(t, x)\|_{\mathfrak{P}} = \{|f| : f(t) \in F(t, x)\} \leq \varphi(t),$$

for a.e. $t \in \mathbf{J}$ and $x \in \mathbf{E}$ with $|x| \leq \varrho$, and

$$\lim_{\varrho \to \infty} \inf \frac{\int_0^b \varphi(t) dt}{\varrho} = \ell < \infty.$$

(H3) *There is a Carathèodory function $\vartheta : \mathbf{J} \times [0, 2\varrho] \to \mathbb{R}_+$ such that*

$$\kappa\left(F(t, G)\right) \leq \vartheta(t, \kappa(G)),$$

a.e. $t \in \mathbf{J}$ and each $G \subset \mathcal{K}$, and the unique solution $\theta \in C(\mathbf{J}, [0, 2\varrho])$ of the inequality

$$\theta(t) \leq 2\left\{\frac{1}{\Gamma(\alpha - 1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds\right) \frac{(u - \tau)^{\alpha - 2}}{k_0(\alpha, u)} \vartheta\left(\tau, \kappa\left(G(\tau)\right)\right)\, d\tau\, du\right\},$$

$t \in \mathbf{J}$,

is $\theta \equiv 0$.

Then the inclusion problem (1.1) possesses at least one solution, provided that

$$\ell < \frac{\Gamma(\alpha) M_{k_0}}{b}, \tag{3.6}$$

where $M_{k_0} := \inf_{t \in \mathbf{J}} |k_0(\alpha, t)| \neq 0$.

Proof. Define the multi-valued map $\mathcal{N} : C(\mathbf{J}, \mathbf{E}) \to \mathfrak{P}(C(\mathbf{J}, \mathbf{E}))$ by

$$(\mathcal{N}x)(t) = \begin{cases} f \in C(\mathbf{J}, \mathbf{E}) : \\ f(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) x_0 \\ \qquad + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w(\tau)\, d\tau\, du, \;\; w \in \mathcal{S}_{F,x}. \end{cases} \tag{3.7}$$

In accordance with Lemma 3.4, the fixed points of $\mathcal{N}$ are solutions to the inclusion problem (1.1). We shall show in five steps that the multi-valued operator $\mathcal{N}$ satisfies all assumptions of Mönch's fixed point theorem (Theorem 2.10) with $\overline{\mathcal{U}} = C(\mathbf{J}, \mathcal{K})$.

Step 1. $\mathcal{N}(x)$ is convex, for any $x \in C(\mathbf{J}, \mathcal{K})$.

For $f_1, f_2 \in \mathcal{N}(x)$, there exist $w_1, w_2 \in \mathcal{S}_{F,x}$ such that for each $t \in \mathbf{J}$, we have

$$\begin{aligned} f_i(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) x_0 \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w_i(\tau)\, d\tau\, du, \;\; i = 1, 2. \end{aligned}$$

Let $0 \le \mu \le 1$. Then, for $t \in \mathbf{J}$,

$$\begin{aligned} &(\mu f_1 + (1-\mu) f_2)(t) = \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) x_0 \\ &+ \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} (\mu w_1 + (1-\mu) w_2)(\tau)\, d\tau\, du. \end{aligned}$$

Since $\mathcal{S}_{F,x}$ is convex (because F has convex values), then $\mu f_1 + (1-\mu) f_2 \in \mathcal{N}(x)$.

Step 2. $\mathcal{N}(G)$ is relatively compact for each compact $G \in \overline{\mathcal{U}}$.

Let $G \in \overline{\mathcal{U}}$ be a compact set and let $\{f_n\}$ be any sequence of elements of $\mathcal{N}(G)$. We show that $\{f_n\}$ has a convergent subsequence by using the Arzelà-Ascoli criterion of non-compactness in $C(\mathbf{J}, \mathcal{K})$. Since $f_n \in \mathcal{N}(G)$, there exist $x_n \in G$ and $w_n \in \mathcal{S}_{F,x_n}$, such that

$$\begin{aligned} f_n(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) x_0 \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w_n(\tau)\, d\tau\, du, \end{aligned}$$

for $n \ge 1$. In view of Theorem 2.11 and the properties of the Kuratowski measure of non-compactness, we have

$$\kappa(\{f_n(t)\}) \le 2\left\{\frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \kappa\left(\left\{\exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds\right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} w_n(\tau) : n \ge 1\right\}\right) d\tau du\right\}. \tag{3.8}$$

On the other hand, since G is compact, the set $\{w_n(\tau) : n \geq 1\}$ is compact. Consequently, $\kappa(\{w_n(\tau) : n \geq 1\}) = 0$ for a.e. $\tau \in \mathbf{J}$. Therefore, $\kappa(\{f_n(t)\}) = 0$ which implies that $\{f_n(t) : n \geq 1\}$ is relatively compact in $\mathcal{K}$ for each $t \in \mathbf{J}$. Furthermore, For each $t_1, t_2 \in \mathbf{J},\ t_1 < t_2$, one obtain that:

$$
\begin{aligned}
&|f_n(t_2) - f_n(t_1)| \\
\leq& \left| \exp\left(- \int_0^{t_2} \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds \right) x_0 - \exp\left(- \int_0^{t_1} \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds \right) x_0 \right| \\
&+ \frac{1}{\Gamma(\alpha-1)} \left| \int_0^{t_2}\int_0^{u} \left[\exp\left(- \int_0^{t_2} \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds \right) - \exp\left(- \int_0^{t_1} \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds \right) \right] \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} w_n(\tau) d\tau du \right| \\
&+ \frac{1}{\Gamma(\alpha-1)} \left| \int_{t_1}^{t_2} \int_0^{u} \exp\left(- \int_0^{t_1} \frac{k_1(\alpha, s)}{k_0(\alpha, s)}\, ds \right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha, u)} w_n(\tau)\, d\tau\, du \right|
\end{aligned}
$$

By applying the mean value theorem to the function $\exp\left(- \int_0^{t} \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\, ds \right)$ on (t_1, t_2), we obtain that

$$
\begin{aligned}
\left| \exp\left(- \int_0^{t_2} \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds \right) - \exp\left(- \int_0^{t_1} \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds \right) \right| &= \left| \frac{k_1(\alpha, \xi)}{k_0(\alpha, \xi)} \exp\left(- \int_0^{\xi} \frac{k_1(\alpha, s)}{k_0(\alpha, s)} ds \right) (t_2 - t_1) \right| \\
&\leq \left| \frac{k_1(\alpha, \xi)}{k_0(\alpha, \xi)} \right| (t_2 - t_1), \quad \forall \xi \in (t_1, t_2).
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
|f_n(t_2) - f_n(t_1)| \leq& \left| \frac{k_1(\alpha, \xi)}{k_0(\alpha, \xi)} \right| |x_0| (t_2 - t_1) \\
&+ \frac{1}{\Gamma(\alpha - 1) M_{k_0}} \left| \frac{k_1(\alpha, \xi)}{k_0(\alpha, \xi)} \right| (t_2 - t_1) \int_0^{t_2} \int_0^{u} (u - \tau)^{\alpha-2} |w_n(\tau)|\, d\tau\, du \\
&+ \frac{1}{\Gamma(\alpha - 1) M_{k_0}} \int_{t_1}^{t_2} \int_0^{u} (u - \tau)^{\alpha-2} |w_n(\tau)|\, d\tau\, du \\
\leq& \left| \frac{k_1(\alpha, \xi)}{k_0(\alpha, \xi)} \right| |x_0| (t_2 - t_1) \\
&+ \frac{1}{\Gamma(\alpha - 1) M_{k_0}} \left| \frac{k_1(\alpha, \xi)}{k_0(\alpha, \xi)} \right| (t_2 - t_1) \int_0^{t_2} \int_0^{u} (u - \tau)^{\alpha-2} \varphi(\tau)\, d\tau\, du \\
&+ \frac{1}{\Gamma(\alpha - 1) M_{k_0}} \int_{t_1}^{t_2} \int_0^{u} (u - \tau)^{\alpha-2} \varphi(\tau)\, d\tau\, du
\end{aligned}
$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. Thus, $\{w_n(\tau) : n \geq 1\}$ is equicontinuous. Hence, $\{w_n(\tau) : n \geq 1\}$ is relatively compact in $C(\mathbf{J}, \mathcal{K})$.

Step 3. The graph of $\mathcal{N}$ is closed.

Let $x_n \to x_*, f_n \in \mathcal{N}(x_n)$, and $f_n \to f_*$. It must be to show that $f_* \in \mathcal{N}(x_*)$. Now, $f_n \in \mathcal{N}(x_n)$ means that there exists $w_n \in \mathcal{S}_{F,x_n}$ such that, for each $t \in \mathbf{J}$,

$$\begin{aligned} f_n(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0 \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w_n(\tau)\,d\tau\,du, \end{aligned}$$

Consider the continuous linear operator $\Theta : L^1(\mathbf{J}, \mathbf{E}) \to C(\mathbf{J}, \mathbf{E})$,

$$\begin{aligned} \Theta(w)(t) \mapsto f_n(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0 \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w_n(\tau)\,d\tau\,du. \end{aligned}$$

It is obvious that $\|f_n - f_*\| \to 0$ as $n \to \infty$. Therefore, in the light of Lemma 2.7, we infer that $\Theta \circ \mathcal{S}_F$ is a closed graph operator. Additionally, $f_n(t) \in \Theta(\mathcal{S}_{F,x_n})$. Since, $x_n \to x_*$, Lemma 2.7 gives

$$\begin{aligned} f_*(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0 \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w(\tau)\,d\tau\,du, \end{aligned}$$

for some $w \in \mathcal{S}_{F,x}$.

Step 4. G is relatively compact in $C(\mathbf{J}, \mathcal{K})$.

Assume that $G \subset \overline{\mathcal{U}}$, $G \subset conv\,(\{0\} \cup \mathcal{N}(G))$, and $\overline{G} = \overline{C}$ for some countable set $C \subset G$. Using a similar approach as in Step 2, one can obtain that $\mathcal{N}(G)$ is equicontinuous. In accordance to $G \subset conv\,(\{0\} \cup \mathcal{N}(G))$, it follows that G is equicontinuous. In addition, since $C \subset G \subset conv\,(\{0\} \cup \mathcal{N}(G))$ and C is countable, then we can find a countable set $\mathbf{P} = \{f_n : n \geq 1\} \subset \mathcal{N}(G)$ with $C \subset conv\,(\{0\} \cup \mathbf{P})$. Thus, there exist $x_n \in G$ and $w_n \in \mathcal{S}_{F,x_n}$ such that

$$\begin{aligned} f_n(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0 \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w_n(\tau)\,d\tau\,du. \end{aligned}$$

In the light of Theorem 2.11 and the fact that $G \subset \overline{C} \subset \overline{conv}\left(\{0\} \cup \mathbf{P}\right)$, we get

$$\kappa\left(G(t)\right) \leq \kappa\left(\overline{C}(t)\right) \leq \kappa\left(\mathbf{P}(t)\right) = \kappa\left(\{f_n(t) : n \geq 1\}\right).$$

By virtue of (3.8) and the fact that $w_n(\tau) \in G(\tau)$, we get

$$\begin{aligned}
&\kappa\left(G(t)\right)\\
&\leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \kappa\left(\left\{\exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w_n(\tau) : n \geq 1\right\}\right) d\tau\, du\right\}\\
&\leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\kappa\left(G(\tau)\right)\,d\tau\,du\right\}\\
&\leq 2\left\{\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}\vartheta\left(\tau,\kappa\left(G(\tau)\right)\right)\,d\tau\,du\right\}
\end{aligned}$$

Also, the function θ given by $\theta(t) = \kappa\left(G(t)\right)$ belongs to $C(\mathbf{J},[0,2\varrho])$. Consequently by **(H3)**, $\theta \equiv 0$, that is $\kappa\left(G(t)\right) = 0$ for all $t \in \mathbf{J}$.

Now, by the Arzelà-Ascoli theorem, G is relatively compact in $C(\mathbf{J},\mathcal{K})$.

Step 5. Let $f \in \mathcal{N}(x)$ with $x \in \overline{\mathcal{U}}$. Since $x(\tau) \leq \varrho$ and **(H2)**, we have $\mathcal{N}(\overline{\mathcal{U}}) \subset \overline{\mathcal{U}}$, because if it is not true, there exists a function $x \in \overline{\mathcal{U}}$ but $\|\mathcal{N}(x)\| > \varrho$ and

$$\begin{aligned}
f(t) &= \exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0\\
&\quad+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)}w(\tau)\,d\tau\,du,
\end{aligned}$$

for some $w \in \mathcal{S}_{F,x}$. On the other hand we have

$$\begin{aligned}
\varrho < \|\mathcal{N}(x)\| &\leq \left|\exp\left(-\int_0^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)x_0\right|\\
&\quad+\frac{1}{\Gamma(\alpha-1)}\int_0^t\int_0^u \left|\exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)}\,ds\right)\right|\frac{(u-\tau)^{\alpha-2}}{|k_0(\alpha,u)|}\,|w(\tau)|\,d\tau\,du\\
&\leq |x_0| + \frac{1}{\Gamma(\alpha-1)M_{k_0}}\int_0^t\int_0^u (u-\tau)^{\alpha-2}\,|w(\tau)|\,d\tau\,du\\
&= |x_0| + \frac{1}{\Gamma(\alpha-1)M_{k_0}}\int_0^t\int_\tau^t (u-\tau)^{\alpha-2}\,|w(\tau)|\,du\,d\tau\\
&= |x_0| + \frac{1}{\Gamma(\alpha)M_{k_0}}\int_0^t (t-\tau)^{\alpha-1}\,|w(\tau)|\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leq |x_0| + \frac{t}{\Gamma(\alpha) M_{k_0}} \int_0^t \varphi(\tau)\, d\tau \\
&\leq |x_0| + \frac{b}{\Gamma(\alpha) M_{k_0}} \int_0^b \varphi(\tau)\, d\tau.
\end{aligned}
$$

Dividing both sides by ϱ and taking the lower limit as $\varrho \to \infty$, we infer that $\frac{b}{\Gamma(\alpha)M_{k_0}}\ell \geq 1$ which contradicts (3.6). Hence $\mathcal{N}(\overline{\mathcal{U}}) \subset \overline{\mathcal{U}}$.

As a consequence of Steps 1-5 together with Theorem 2.10, we infer that $\mathcal{N}$ possesses a fixed point $x \in C(\mathbf{J}, \mathcal{K})$ which is a solution of the inclusion problem (1.1). $\square$

4. Example

Consider the fractional differential inclusion

$$
\begin{cases}
{}^{PC}_{\ 0}\mathcal{D}_t^{\frac{1}{2}} x(t) \in F\left(t, x(t)\right), & \text{a.e. on } [0,1], \\
x(0) = 0,
\end{cases}
\tag{4.1}
$$

where $\alpha = \frac{1}{2}, b = 1, x_0 = 0$, and $F : [0,1] \times \mathbb{R} \to \mathfrak{P}(\mathbb{R})$ is a multi-valued map given by

$$
x \mapsto F\left(t, x\right) = \left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3\right).
$$

For $f \in F$, one has

$$
|f| = \max\left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3\right) \leq 9, \quad x \in \mathbb{R}.
$$

Thus

$$
\begin{aligned}
\|F(t,x)\|_{\mathfrak{P}} &= \{|f| : f \in F(t,x)\} \\
&= \max\left(e^{-|x|} + \sin t, 3 + \frac{|x|}{1+x^2} + 5t^3\right) \leq 9 = \varphi(t),
\end{aligned}
$$

for $t \in [0,1]$, $x \in \mathbb{R}$. Obviously, F is compact and convex valued, and it is upper semi-continuous.

Furthermore, for $(t,x) \in [0,1] \times \in \mathbb{R}$ with $|x| \leq \varrho$, one has

$$
\lim_{\varrho \to \infty} \inf \frac{\int_0^1 \varphi(t) dt}{\varrho} = 0 = \ell.
$$

Therefore, for a suitable M_{k_0}, the condition (3.6) implies that

$$
\frac{\Gamma(1/2) M_{k_0}}{b} = M_{k_0}\sqrt{\pi} > 0.
$$

Finally, we assume that there exists a Carathèodory function $\vartheta : [0,1] \times [0, 2\varrho] \to \mathbb{R}_+$ such that

$$\kappa\left(F(t,G)\right) \leq \vartheta(t, \kappa(G)),$$

a.e. $t \in [0,1]$ and each $G \subset \mathcal{K} = \{x \in \mathbb{R} : |x| \leq \varrho\}$, and the unique solution $\theta \in C([0,1],[0,2\varrho])$ of the inequality

$$\theta(t) \leq 2\left\{ \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_0^u \exp\left(-\int_u^t \frac{k_1(\alpha,s)}{k_0(\alpha,s)} ds \right) \frac{(u-\tau)^{\alpha-2}}{k_0(\alpha,u)} \vartheta\left(\tau, \kappa\left(G(\tau)\right)\right) d\tau du \right\}, \quad t \in \mathbf{J},$$

is $\theta \equiv 0$.

Hence all the assumptions of Theorem 3.5 hold true and we infer that the inclusion problem (4.1) possesses at least one solution on $[0,1]$.

5. Conclusions

In this paper, we extend the investigation of fractional differential inclusions to the case of hybrid Caputo-proportional fractional derivatives in Banach space. Based on the set-valued version of Mönch fixed point theorem together with the Kuratowski measure of non-compactness, the existence theorem of the solutions for the proposed inclusion problem is founded. An clarified example is suggested to understand the theoretical finding. Furthermore, the obtained results in this paper can be employed in future work in the sense of the generalized fractional derivative (GFD) definition which was recently proposed in [2, 3]. This new definition overcomes some issues associated with some conformable derivative and some other fractional derivatives.

References

[1] M.I. Abbas and S. Hristova, *Existence results of nonlinear generalized proportional fractional differential inclusions via the diagonalization technique*, AIMS Mathematics 6 (11) (2021) 12832–12844.

[2] M. Abu-Shady, Mohammed K. A. Kaabar, *A Generalized Definition of the Fractional Derivative with Applications*, Mathematical Problems in Engineering 2021, Article ID 9444803.

[3] M. Abu-Shady, M.K.A. Kaabar, *A Novel Computational Tool for the Fractional-Order Special Functions Arising from Modeling Scientific Phenomena via Abu-Shady-Kaabar Fractional Derivative*, Computational and Mathematical Methods in Medicine 2022, Article ID 2138775.

[4] B. Ahmad, A. Alsaedi, S.K. Ntouyas, H.H. Al-Sulami, *On neutral functional differential inclusions involving Hadamard fractional derivatives*, Mathematics 7 (11) (2019) 1–13, Article ID:1084.

[5] B. Alqahtani, S. Abbas, M. Benchohra, S.S. Alzaid, *Fractional q-difference inclusions in Banach spaces*, Mathematics 8 (91) (2020) 91, 1–12, DOI: 10.3390/math8010091.

[6] D.R. Anderson, D.J. Ulness, *Newly defined conformable derivatives*, Advances in Dynamical Systems and Applications 10 (2) (2015) 109–137.

[7] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific: Singapore, 2012.

[8] D. Baleanu, A. Fernandez, A. Akgül, *On a fractional operator combining proportional and classical differintegrals*, Mathematics 8 (3) (2020) 360, 1–13.

[9] J. Banaś, K. Goebel, *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.

[10] J. Banaś, T. Zając, *On a measure of noncompactness in the space of regulated functions and its applications*, Adv. Nonlinear Anal. 8 (2019) 1099–1110.

[11] M. Benchohra, J.R. Graef, N. Guerraiche, S. Hamani, *Nonlinear boundary value problems for fractional differential inclusions with Caputo-Hadamard derivatives on the half line*, AIMS Mathematics 6 (6) (2021) 6278–6292.

[12] M. Benchohra, S. Hamani, *Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative*, Topol. Meth. Nonlinear Anal. 32 (1) (2008) 115–130.

[13] M. Benchohra, N. Hamidi, J.J. Nieto, *Existence of solutions to differential inclusions with fractional order and impulses*, Electron. J. Qual. Theory Differ. Equ. 80 (2010) 1–18.

[14] M. Benchohra, J. Henderson, D. Seba, *Boundary value problems for fractional differential inclusions in Banach space*, Fract. Differ. Calc. 2 (1) (2012) 99–108.

[15] A. Das, B. Hazarika, V. Parvaneh, M. Mursaleen, *Solvability of generalized fractional order integral equations via measures of noncompactness*, Math. Sci. (2021) DOI:10.1007/s40096-020-00359-0.

[16] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.

[17] K. Diethelm, A.D. Freed, *On the solution of nonlinear fractional-order differential equations used in the modeling of viscoelasticity*, in Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, F. Keil, W. Mackens, H. Voss, and J. Werther, Eds., Springer-Verlag, Heidelberg, 1999, 217–224.

[18] J.R. Graef, N. Guerraiche, S. Hamani, *Boundary value problems for fractional differential inclusions with Hadamard type derivatives in Banach spaces*, Stud. Univ. Babeş-Bolyai Math. 62 (4) (2017) 427–438.

[19] B. Hazarika, R. Arab, M. Mursaleen, *Applications of measure of moncompactness and operator type contraction for existence of solution of functional integral equations*, Complex Anal. Oper. Theo. (2019), doi:10.1007/s11785-019-00933-y.

[20] H. P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. 7 (1983) 1351–1371.

[21] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[22] M. Houas, F. Martinez, M.E. Samei, M.K.A. Kaabar, *Uniqueness and Ulam-Hyers-Rassias stability results for sequential fractional pantograph q- differential equations*, Journal of Inequalities and Applications 2022 (1) (2022) 1–24.

[23] M.K.A. Kaabar, A. Refice, M.S. Souid, F. Martinez, S. Etemad, Z. Siri, S. Rezapour, *Existence and UHR stability of solutions to the implicit nonlinear FBVP in the variable order settings*, Mathematics 9 (14) (2022) 1693, 1–17.

[24] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B. V., Amsterdam, 2006.

[25] A. Lasota, Z. Opial, *An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. 13 (1965) 781–786.

[26] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World Scientific Publishing Company: Singapore; Hackensack, NJ, USA, London, UK; Hong Kong, China, 2010.

[27] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley-Interscience, John-Wiley and Sons: New York, NY, USA, 1993.

[28] A. Nouara, A. Amara, E. Kaslik, S. Etemad, S. Rezapour, F. Martinez, M.K.A. Kaabar, *A study on multiterm hybrid multi-order fractional boundary value problem coupled with its stability analysis of Ulam-Hyers type*, Advances in Difference Equations 2021 (1) (2021) 1–28.

[29] N. Nyamoradi, D. Baleanu, R. Agarwal, *On a multipoint boundary value problem for a fractional order differential inclusion on an infinite interval*, Adv. Math. Phys. 2013 (2013), Article ID:823961.

[30] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[31] A. Salem, H.M. Alshehri, L. Almaghamsi, *Measure of noncompactness for an infinite system of fractional Langevin equation in a sequence space*, Adv. Difference Equa. 132 (2021), DOI:10.1186/s13662-021-03302-2.

[32] S. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Drivatives*, Gordon and Breach Science Publishers, Longhorne, PA, 1993.

[33] H.M. Srivastava, K.M. Saad, *Some new models of the time-fractional gas dynamics equation*, Adv. Math. Models Appl. 3 (1) (2018) 5–17.

DOI: 10.7862/rf.2022.1

Mohamed I. Abbas
email: `miabbas@alexu.edu.eg`
ORCID: 0000-0002-3803-8114
Department of Mathematics and Computer Science, Faculty of Science,
Alexandria University
Alexandria 21511
EGYPT

Received 15.02.2022 *Accepted 13.09.2022*