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A Contribution on Real and Complex Convexity in Several Complex Variables

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Abstract: Let $f, g : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions. Define $u(z,w) = |w - f(z)|^4 + |w - g(z)|^4$, $v(z,w) = |w - f(z)|^2 + |w - g(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. A comparison between the convexity of u and v is obtained under suitable conditions.
Now consider four holomorphic functions $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$. We prove that $F = |\varphi_1 - g_1|^2 + |\varphi_2 - g_2|^2$ is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$ if and only if $n = m = 1$ and φ_1, φ_2, g_1, g_2 are affine functions with $(\varphi_1' g_2' - \varphi_2' g_1') \neq 0$.
Finally, it is shown that the product of four absolute values of pluriharmonic functions is plurisubharmonic if and only if the functions satisfy special conditions as well.

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1. Introduction

Convex functions recently are studied in complex analysis because they appear in the theory of holomorphic functions, plurisubharmonic (psh) functions, currents, Lelong numbers, extension problems, holomorphic representation theory (see [2], [5], [6], [7], [8], [10], [11], [13], [14], [15], [16], [17] and [19]).
It is worth mentioning that an interesting relation between convex and plurisubharmonic functions has been obtained in [2].
Several papers appeared recently to this topic, let us mention [2], [3], [5], [6], [15], [19] and the monographs [11], [14], [19] and more recently [5].

Let $n \geq 1$. We can construct a C^∞ strictly psh function F defined on $\mathbb{C}^n \times \mathbb{C}$, such that F is not convex (and not concave) on each Euclidean not empty open ball subset in $\mathbb{C}^n \times \mathbb{C}$. For instance,

$$F(z,w) = |w - e^{\overline{z_1}}|^2 + ... + |w - e^{\overline{z_n}}|^2, \text{ for } z = (z_1, ..., z_n) \in \mathbb{C}^n, \ w \in \mathbb{C}.$$

Moreover, for the case of one complex variable, let $\lambda(z) = 2x^2 - y^2$, $z = (x + iy) \in \mathbb{C}$, $x = \mathrm{Re}(z)$. Then λ is a C^∞ strictly sh function on $\mathbb{C}$, while λ is not convex (respectively not concave) at each point of $\mathbb{C}$.
This proves that the new class of functions, consisting of convex and strictly psh functions, is well defined because we can not compare the two families (convex functions) and (convex and strictly psh functions).
Now thanks to [2], we know the holomorphic representation of each holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ under the suitable condition of the convexity of its modulus.

Let $\delta \in [1, +\infty[$. We have the following observation.
Put $K(z,w) = |w - f(z)|^\delta$ and $H(z,w) = |w - f(z)|$, for $(z,w) \in \mathbb{C}^2$, where $f : \mathbb{C} \to \mathbb{C}$ is holomorphic. Assume that K is convex on $\mathbb{C}^2$ and $\delta > 1$. Then H is convex on $\mathbb{C}^2$ and we have H^s is convex on $\mathbb{C}^2$, for each $s \in [1, +\infty[$ independently of δ and conversely.
Now let $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ be two holomorphic functions and $s \in \mathbb{N} \backslash \{0\}$. Define $K_{2s}(z,w) = |w - f_1(z)|^{2s} + |w - f_2(z)|^{2s}$, for $(z,w) \in \mathbb{C}^2$. By theorem 10, we have that K_4 is convex on $\mathbb{C}^2$ implies that K_2 is convex on $\mathbb{C}^2$. But the converse is not true. For instance, let $f_1(z) = z^4$, $f_2(z) = -z^4$, $z \in \mathbb{C}$. Then K_2 is convex on $\mathbb{C}^2$. But K_4 is not convex on $\mathbb{C}^2$. This remark leads to the following problem.

Let $N \in \mathbb{N} \backslash \{0, 1\}$ and $F_1, ..., F_N : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions. Define

$$\psi_\delta(z,w) = |w - F_1(z)|^\delta + ... + |w - F_N(z)|^\delta, \text{ for } (z,w) \in \mathbb{C}^n \times \mathbb{C}.$$

Suppose that ψ_δ is convex on $\mathbb{C}^n \times \mathbb{C}$.

Firstly, for the study of the convexity of ψ_δ, we observe that we study separately the following two cases.
Case 1. $\delta \in [1, +\infty[\backslash\{2\}$.
Case 2. $\delta = 2$.
Is it true that $\delta \in [1, +\infty[\backslash\{2\}$, implies that $F_1, ..., F_N$ are affine functions?
Recall that for $\delta = 2$, there exists several cases where ψ_2 is convex on $\mathbb{C}^n \times \mathbb{C}$, but $F_1, ..., F_N$ are not affine functions.
Moreover, for $N = 2$, by a limiting argument and a specific holomorphic differential equation, we prove that ψ_1 is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if F_1 and F_2 are affine functions. Indeed, ψ_{2k} is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if F_1 and F_2 are affine functions, for $k \in \mathbb{N} \backslash \{0, 1\}$.

The paper is organized as follows. In section 2, we shall use an elementary holomorphic differential equation in the proofs of the following two technical questions.
Let $A_1, A_2 \in \mathbb{C}$ and $n, m \in \mathbb{N} \backslash \{0\}$. Characterize exactly all the 3 holomorphic functions $\varphi : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that u is convex (respectively convex and strictly plurisubharmonic) on $\mathbb{C}^n \times \mathbb{C}^m$, where

$$u(z,w) = |A_1\varphi(w) - g_1(z)|^2 + |A_2\varphi(w) - g_2(z)|^2, \text{ for } (z,w) \in \mathbb{C}^n \times \mathbb{C}^m.$$

In this case find the expressions of φ, g_1 and g_2.
Moreover, find all the three holomorphic functions $\varphi : \mathbb{C}^m \to \mathbb{C}$ and $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ such that v is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$, where

$$v(z,w) = |A_1\varphi(w) - \overline{f_1}(z)|^2 + |A_2\varphi(w) - \overline{f_2}(z)|^2,\ (z,w) \in \mathbb{C}^n \times \mathbb{C}^m.$$

We prove that we have a great differences between the 2 classes of functions defined similar as u and v.

Now let $k_1, k_2 : G \to \mathbb{C}^t$ be two holomorphic functions. Then the functions $\parallel k_1 + \overline{k_2} \parallel^2$ and $(\parallel k_1 + \lambda \parallel^2 + \parallel k_2 + \delta \parallel^2)$ have the same hermitian Levi form on G, where G is a domain of $\mathbb{C}^s$, $\lambda, \delta \in \mathbb{C}^t$ and $s, t \in \mathbb{N}\backslash\{0\}$.
For the applications, we can see the proof of theorem 4, corollary 1, theorem 5 and others.

In section 3, we consider the following problems.
Problem 1. Let $n, m \geq 1$. Find all the 4 holomorphic functions $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that $\psi = |\varphi_1 - g_1|^2 + |\varphi_2 - g_2|^2$ is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$.
Problem 2. Characterize all the holomorphic functions $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that $\psi = |\varphi_1 - g_1|^2 + |\varphi_2 - g_2|^2$ is convex and strictly psh (respectively convex) on $\mathbb{C}^n \times \mathbb{C}^m$.

Before stating it, we can study the analysis question. Find all the holomorphic functions $\varphi_1, \varphi_2, \psi_1, \psi_2 : \mathbb{C}^m \to \mathbb{C}$ and $f_1, f_2, g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$, such that u_1 and u_2 are convex and $u = (u_1 + u_2)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$. Where $u_1(z,w) = |\varphi_1(w) - f_1(z)|^2 + |\varphi_2(w) - f_2(z)|^2$, $u_2(z,w) = |\psi_1(w) - g_1(z)|^2 + |\psi_2(w) - g_2(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.

In section 4, we use an algebraic method to mainly focus on properties of the new structure (convex and strictly psh) and their relations with the holomorphic representation theory.

In section 5 we study the product of several absolute values of pluriharmonic (prh) functions and some auxiliary results are proved.
Let U be a domain of $\mathbb{R}^d$, $(d \geq 2)$. Put sh(U) the set of all subharmonic functions on U.
For $f : U \to \mathbb{C}$ be a function, $|f|$ is the modulus of f. For $N \geq 1$ and $h = (h_1, ..., h_N)$, where $h_1, ..., h_N : U \to \mathbb{C}$, $\parallel h \parallel = (|h_1|^2 + ... + |h_N|^2)^{\frac{1}{2}}$.
Let $g : D \to \mathbb{C}$ be an analytic function, D is a domain of $\mathbb{C}$. We denote $\frac{\partial^m g}{\partial z^m}$ the holomorphic derivative of g of order m, for all $m \in \mathbb{N}\backslash\{0\}$.
If $\xi = (\xi_1, ..., \xi_n) \in \mathbb{C}^n$, and $z = (z_1, \ldots, z_n \in \mathbb{C}^n$ we write $< z/\xi >= z_1\overline{\xi_1} + ... + z_n\overline{\xi_n}$ and $B(\xi, r) = \{\zeta \in \mathbb{C}^n / \parallel \zeta - \xi \parallel < r\}$ for $r > 0$, where $\sqrt{< \xi/\xi >} = \parallel \xi \parallel$ is the Euclidean norm of ξ. The Lebesgue measure on $\mathbb{C}^n$ is denoted by m_{2n} and $C^k(U) = \{\varphi : U \to \mathbb{C}\ /\ \varphi \text{ is a function of class } C^k \text{ on } U\}$, $k \in \mathbb{N} \cup \{\infty\}\backslash\{0\}$.
Let D be a domain of $\mathbb{C}^n$, $(n \geq 1)$. An usual psh(D) and prh(D) are respectively the classes of plurisubharmonic and pluriharmonic functions on D. For all $a \in \mathbb{C}, |a|$ is the modulus of a, Re(a) is the real part of a and $D(a,r) = \{z \in \mathbb{C}\ /\ |z - a| < r\}$ for $r > 0$.

For the study of properties and extension problems of analytic and plurisubharmonic functions we cite the references [1], [6], [7], [8], [9], [10], [12], [13], [15], [16] and [17]. For the study of convex functions in complex convex domains, we cite [5], [11],

[14], [2] and [19].
For the theory of $n-$ subharmonic functions we cite [18].

2. A family of analytic functions and the holomorphic representation theory

We have

Lemma 1. *Let $g = (g_1, ..., g_N), f = (f_1, ..., f_N) : D \to \mathbb{C}^N$ be two holomorphic functions, $N \geq 1$, D is a domain of $\mathbb{C}^n$, $n \geq 1$ and $a, b \in \mathbb{C}^N$. Then $\parallel f + \overline{g} \parallel^2$ and $(\parallel f + a \parallel^2 + \parallel g + b \parallel^2)$ have the same hermitian Levi form on D. On the other hand, let $u : D \to \mathbb{R}$ be a function of class C^2. Define $u_1 = (u + \parallel f + \overline{g} \parallel^2)$, $u_2 = (u + \parallel f + a \parallel^2 + \parallel g + b \parallel^2)$.
Then u_1 and u_2 are functions of class C^2 on D and we have the assertion.
The function u_1 is strictly psh on D if and only if u_2 is strictly psh on D.
(Observe that if $N < n$, then $\parallel g \parallel^2$ is not strictly psh at each point of D).*

Proof. We have $\parallel f + \overline{g} \parallel^2 = |f_1 + \overline{g_1}|^2 + ... + |f_N + \overline{g_N}|^2 = |f_1|^2 + |g_1|^2 + ... + |f_N|^2 + |g_N|^2 + \sum_{j=1}^{N}(g_j f_j + \overline{g_j}\overline{f_j}) = \parallel g \parallel^2 + \parallel f \parallel^2 + \sum_{j=1}^{N}(g_j f_j + \overline{g_j}\overline{f_j})$.

Since $(g_j f_j + \overline{g_j}\overline{f_j})$ is prh on D, then $\sum_{j=1}^{N}(g_j f_j + \overline{g_j}\overline{f_j})$ is prh on D.

Consequently, $\parallel f + \overline{g} \parallel^2$ and $(\parallel f + a \parallel^2 + \parallel g + b \parallel^2)$ have the same hermitian Levi form on D.

By [4], we have

Theorem 1. *Let $\varphi : \mathbb{C}^m \to \mathbb{C}$ be a holomorphic nonconstant function, $m \geq 1$. Given $A_1, A_2 \in \mathbb{C}\backslash\{0\}$ and $n \geq 1$.
The following conditions are equivalent
(I) There exists 2 holomorphic functions $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that u is convex on $\mathbb{C}^n \times \mathbb{C}^m$, $u(z, w) = |A_1\varphi(w) - g_1(z)|^2 + |A_2\varphi(w) - g_2(z)|^2$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$;
(II) There exists $c \in \mathbb{C}$ such that $|\varphi + c|^2$ is convex on $\mathbb{C}^m$.*

Now in all of this section, $(A_1, A_2) \in \mathbb{C}^2$. Let $\varphi : \mathbb{C}^m \to \mathbb{C}$ be a holomorphic nonconstant function, $m \geq 1$. Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be 2 holomorphic functions, $n \geq 1$. Define $u(z, w) = |A_1\varphi(w) - g_1(z)|^2 + |A_2\varphi(w) - g_2(z)|^2$, $u_1(z, w) = |A_1\varphi(w) - \overline{g_1}(z)|^2 + |A_2\varphi(w) - \overline{g_2}(z)|^2$, $u_2 = u + u_1$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$. $v(z, w) = |A_1\overline{\varphi}(w) - g_1(z)|^2 + |A_2\overline{\varphi}(w) - g_2(z)|^2$, $v_1(z, w) = |A_1\overline{\varphi}(w) - \overline{g_1}(z)|^2 + |A_2\overline{\varphi}(w) - \overline{g_2}(z)|^2$ and $v_2 = v + v_1$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$. We have

Theorem 2. *Assume that $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$. The following conditions are equivalent
(I) u is convex on $\mathbb{C}^n \times \mathbb{C}^m$;
(II) φ is an affine function on $\mathbb{C}^m$, or φ is not affine and there exists $c \in \mathbb{C}$ such that $|\varphi + c|^2$ is convex on $\mathbb{C}^m$ and we have the following cases.
Case 1. The function φ is affine on $\mathbb{C}^m$.*

Then we have the representation

$$\begin{cases} g_1(z) = A_1(< z/\lambda_1 > +\mu_1) + \overline{A_2}\varphi_1(z) \\ g_2(z) = A_2(< z/\lambda_1 > +\mu_1) - \overline{A_1}\varphi_1(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, *where* $\lambda_1 \in \mathbb{C}^n$, $\mu_1 \in \mathbb{C}$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ *is analytic,* $|\varphi_1|^2$ *is convex on* $\mathbb{C}^n$.
Case 2. φ *is not affine on* $\mathbb{C}^n$.
In this case there exists $c \in \mathbb{C}$ *such that* $|\varphi + c|^2$ *is convex on* $\mathbb{C}^m$. *Then we have the representation*

$$\begin{cases} g_1(z) = A_1 c + \overline{A_2}\varphi_1(z) \\ g_2(z) = A_2 c - \overline{A_1}\varphi_1(z) \end{cases}$$

for every $z \in \mathbb{C}^n$, *where* $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ *is analytic,* $|\varphi_1|^2$ *is convex on* $\mathbb{C}^n$.

We can discuss the cases $(A_1, A_2 \in \mathbb{C}\backslash\{0\})$, or $(A_1 \in \mathbb{C}\backslash\{0\}, A_2 = 0)$, or $(A_1 = 0, A_2 \in \mathbb{C}\backslash\{0\})$.
This theorem motivates the following questions. Find all the holomorphic representation of the analytic functions $f_1, f_2, f_3 : \mathbb{C}^n \to \mathbb{C}$, such that ψ is convex on $\mathbb{C}^n \times \mathbb{C}$. $\psi(z, w) = |B_1 w - f_1(z)|^2 + |B_2 w - f_2(z)|^2 + |B_3 w - f_3(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, where $(B_1, B_2, B_3) \in \mathbb{C}^3\backslash\{0\}$.
Indeed, for instance, in harmonic analysis and convex analysis, actually the following question appeared naturally.
Find all the representation of the harmonic functions $F_1, F_2, F_3 : \mathbb{C} \to \mathbb{C}$, such that ψ_1 is convex and strictly $2-$sh on $\mathbb{C}^2$. Where $\psi_1(z, w) = |w - F_1(z)|^2 + |w - F_2(z)|^2 + |w - F_3(z)|^2$, $(z, w) \in \mathbb{C}^2$. (We study here functions on harmonic representation theory). Define $\psi_0(z, w) = |w - F_1(z)|^2 + |w - F_2(z)|^2$, for $(z, w) \in \mathbb{C}^2$. If we choose F_3 is affine on $\mathbb{C}$ and ψ_0 is convex and strictly $2-$ sh on $\mathbb{C}^2$, then we have a family of harmonic functions which satisfy the above condition.
The proof of this theorem is obvious and analogous to the proof of the following.

Theorem 3. *The following conditions are equivalent*
(I) u *is convex and strictly psh on* $\mathbb{C}^n \times \mathbb{C}^m$;
(II) $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$, $n = m = 1$, *there exists* $c \in \mathbb{C}$ *such that* $|\varphi + c|^2$ *is convex on* $\mathbb{C}$ *and we have the following cases.*
Case 1. $A_1 A_2 \neq 0$. *Then*

$$\begin{cases} g_1(z) + A_1 c = A_1(az + b) + \overline{A_2}\psi(z) \\ g_2(z) + A_2 c = A_2(az + b) - \overline{A_1}\psi(z) \end{cases}$$

for each $z \in \mathbb{C}$, *where* $a, b \in \mathbb{C}$, $\psi : \mathbb{C} \to \mathbb{C}$ *is holomorphic,* $|\psi|$ *is convex with* $|\psi'| > 0$ *and* $|\varphi'| > 0$ *on* $\mathbb{C}$.
Case 2. $A_1 \neq 0$ *and* $A_2 = 0$.
If φ *is affine and nonconstant on* $\mathbb{C}$. *Then we have the representation*

$$\begin{cases} g_1(z) = A_1(\lambda z + \mu) \\ g_2(z) = -\overline{A_1}\varphi_2(z) \end{cases}$$

for each $z \in \mathbb{C}$, where $\lambda \in \mathbb{C}\backslash\{0\}$, $\mu \in \mathbb{C}$, $\varphi_2 : \mathbb{C} \to \mathbb{C}$ is analytic, $|\varphi_2|^2$ is convex and strictly subharmonic (sh) on $\mathbb{C}$.
If φ is not affine on $\mathbb{C}$. Then we have the representation

$$\begin{cases} g_1(z) = -A_1 c \\ g_2(z) = -\overline{A_1}\varphi_3(z) \end{cases}$$

for every $z \in \mathbb{C}$, where $\varphi_3 : \mathbb{C} \to \mathbb{C}$ is analytic, $|\varphi_3|^2$ is convex and strictly subharmonic on $\mathbb{C}$. In this situation we have $\varphi(w) = e^{(aw+b)} - c$, for each $w \in \mathbb{C}$, with $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$.
Case 3. $A_1 = 0$ and $A_2 \neq 0$. (Obviously analogous to case 2).

Proof. (I) implies (II). We choose the following proof which have technical applications in the case when we study the convexity of the function F, $F(z,w) = |w - \psi_1(z)|^{2N} + |w - \psi_2(z)|^{2N}$, $N \in \mathbb{N}$, $N \geq 2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, $\psi_1, \psi_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. In this situation we prove that ψ_1 and ψ_2 have analytic representations using the holomorphic differential equation $k''(k+\lambda) = \gamma(k')^2$, where $k : \mathbb{C} \to \mathbb{C}$ is a holomorphic function and $\lambda, \gamma \in \mathbb{C}$.
If $(A_1, A_2) = (0,0)$, then u is independent of w. Thus u is not strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$. A contradiction.
The case where $A_1 \neq 0$ and $A_2 = 0$.
Since $u(0,.)$ is strictly psh on $\mathbb{C}^m$. Then the function $|A_1\varphi - g_1(0)|^2$ is strictly psh on $\mathbb{C}^m$. Thus by lemma 1, $m = 1$. Since $u(.,0)$ is convex on $\mathbb{C}$, then $|\varphi - \frac{g_1(0)}{A_1}|^2$ is convex and strictly sh on $\mathbb{C}$. Put $c = -\frac{g_1(0)}{A_1}$. Now $|\varphi + c|^2$ is convex and strictly sh on $\mathbb{C}$, therefore, by Abidi [2], we have
$\varphi(w) = aw + b$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$, or
$\varphi(w) = e^{(a_1 w + b_1)} - c$, for all $w \in \mathbb{C}$, with $a_1 \in \mathbb{C}\backslash\{0\}$ and $b_1 \in \mathbb{C}$.
If $\varphi(w) = aw + b$, $\forall w \in \mathbb{C}$.
Then for each fixed $w_0 \in \mathbb{C}$, the function $u(.,w_0)$ is convex on $\mathbb{C}^n$.
Therefore,

$$|-\sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)[\overline{A_1}(\overline{aw_0 + b}) - \overline{g_1}(z)]\alpha_j\alpha_k + \sum_{j,k=1}^{n} \frac{\partial^2 g_2}{\partial z_j \partial z_k}(z)\overline{g_2}(z)\alpha_j\alpha_k|$$
$$\leq |\sum_{j=1}^{n} \frac{\partial g_1}{\partial z_j}(z)\alpha_j|^2 + |\sum_{j=1}^{n} \frac{\partial g_2}{\partial z_j}(z)\alpha_j|^2,$$

for each $z \in \mathbb{C}^n$, $w_0 \in \mathbb{C}$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n$.
Since the right hand side of the above inequality is independent of $w_0 \in \mathbb{C}$, it follows that for every fixed $z \in \mathbb{C}^n$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k = 0, \text{ for all } \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{C}^n.$$

Therefore g_1 is affine on $\mathbb{C}^n$.
Put $g_1(z) = A_1(< z/\gamma > +\delta)$, for $z \in \mathbb{C}^n$, where $\gamma \in \mathbb{C}^n$ and $\delta \in \mathbb{C}$.

Let $T : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n \times \mathbb{C}$, $T(z,w) = (z, w + \frac{g_1(z)}{A_1 a} - \frac{\delta}{a})$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
Note that T is a $\mathbb{C}$ linear bijective transformation on $\mathbb{C}^n \times \mathbb{C}$.
Since u is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$, then ψ is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$, where $\psi(z,w) = u \circ T(z,w) = |A_1(aw + b - \delta)|^2 + |g_2(z)|^2$, for every $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
But ψ is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}$, then $|g_2|^2$ is convex and strictly psh on $\mathbb{C}^n$. Thus $n = 1$.
Put $g_2(z) = -\overline{A_1}\varphi_2(z)$, for $z \in \mathbb{C}$ (φ_2 is analytic on $\mathbb{C}$). Thus $|\varphi_2|^2$ is convex and strictly sh on $\mathbb{C}$.
(II) implies (I). Obvious.

Question. Let $B_1, B_2 \in \mathbb{C}\backslash\{0\}$. For $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$, define $\psi(z,w) = |B_1 w - f_1(z)|^2 + |B_2 w - f_2(z)|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Find all the pluriharmonic (respectively $n-$harmonic) functions $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$, such that ψ is convex (respectively convex and strictly $n-$subharmonic) on $\mathbb{C}^n \times \mathbb{C}$.

Theorem 4. *The following conditions are equivalent*
(I) u_1 is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$;
(II) $m = 1$, $n \in \{1,2\}$, $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$, there exists $c \in \mathbb{C}$ such that $|\varphi + c|^2$ is convex on $\mathbb{C}$ and we have the following cases.
Case 1. For all $w \in \mathbb{C}$, $\varphi(w) = aw + b$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$.
We have the representation

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda_1 > +\mu_1) + A_2\varphi_1(z) \\ g_2(z) = \overline{A_2}(< z/\lambda_1 > +\mu_1) - A_1\varphi_1(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, where $\lambda_1 \in \mathbb{C}^n$, $\mu_1 \in \mathbb{C}$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ is analytic, $|\varphi_1|^2$ is convex on $\mathbb{C}^n$, such that
$(n = 1, \lambda_1 \neq 0)$, *or*
$(n = 1, \lambda_1 = 0, \frac{\partial \varphi_1}{\partial z}(z) \neq 0,$ for each $z \in \mathbb{C})$, *or*
$n = 2, (\lambda_1, (\overline{\frac{\partial \varphi_1}{\partial z_1}(z)}, \overline{\frac{\partial \varphi_1}{\partial z_2}(z)}))$ is a basis of the complex vector space $\mathbb{C}^2$,
for each $z = (z_1, z_2) \in \mathbb{C}^2$.
Case 2. For every $w \in \mathbb{C}$, $\varphi(w) = e^{(aw+b)} - c$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$.
Then $n = 1$ and we have the representation

$$\begin{cases} g_1(z) = -\overline{A_1}\overline{c} + A_2\psi_1(z) \\ g_2(z) = -\overline{A_2}\overline{c} - A_1\psi_1(z) \end{cases}$$

for each $z \in \mathbb{C}$, where $\psi_1 : \mathbb{C} \to \mathbb{C}$ is analytic, $|\psi_1|^2$ is convex and strictly sh on $\mathbb{C}$.

The proof follows from the above 3 theorems and lemma 1.
We have

Corollary 1. *The following conditions are equivalent*
(I) u is convex on $\mathbb{C}^n \times \mathbb{C}^m$ and u_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$;
(II) u is convex on $\mathbb{C}^n \times \mathbb{C}^m$ and u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$;
(III) $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$, $m = 1$, $n \in \{1,2\}$, there exists $c \in \mathbb{C}$ such that $|\varphi + c|^2$ is convex and strictly sh on $\mathbb{C}$ and we have the following 2 cases.

Case 1. For all $w \in \mathbb{C}$ $\varphi(w) = aw + b$, $(a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C})$.
Then we have the holomorphic representation

$$\begin{cases} g_1(z) = A_1(< z/\lambda_1 > +\mu_1) + \overline{A_2}\varphi_1(z) \\ g_2(z) = A_2(< z/\lambda_1 > +\mu_1) - \overline{A_1}\varphi_1(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, *where* $\lambda_1 \in \mathbb{C}^n$, $\mu_1 \in \mathbb{C}$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ *is analytic,* $|\varphi_1|^2$ *is convex on* $\mathbb{C}^n$, *such that*
$(n = 1, \lambda_1 \neq 0)$, *or* $(n = 1, \lambda_1 = 0, \frac{\partial \varphi_1}{\partial z}(z) \neq 0,$ for each $z \in \mathbb{C})$, *or*
$(n = 2$ and $(\lambda_1, (\overline{\frac{\partial \varphi_1}{\partial z_1}}(z), \overline{\frac{\partial \varphi_1}{\partial z_2}}(z)))$ is a basis of the complex vector space $\mathbb{C}^2$,
for every $z = (z_1, z_2) \in \mathbb{C}^2)$.
Case 2. For all $w \in \mathbb{C}$, $\varphi(w) = e^{(aw+b)} - c$, *where* $a \in \mathbb{C}\backslash\{0\}$ *and* $b \in \mathbb{C}$.
Then $n = 1$ *and we have the holomorphic representation*

$$\begin{cases} g_1(z) = -A_1 c + \overline{A_2}\psi_1(z) \\ g_2(z) = -A_2 c - \overline{A_1}\psi_1(z) \end{cases}$$

for every $z \in \mathbb{C}$, *where* $\psi_1 : \mathbb{C} \to \mathbb{C}$ *is analytic,* $|\psi_1|^2$ *is convex and strictly sh on* $\mathbb{C}$.

Proof. (I) implies (III). Note that u, u_1 and u_2 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}^m$. We have
u_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$ if and only if u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$.
Assume that $(A_1, A_2) = (0, 0)$. Then u_1 is independent of $w \in \mathbb{C}^m$ and u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$. A contradiction.
Consequently, $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$.
Define $u_3(z, w) = (|A_1|^2 + |A_2|^2)|\varphi(w)|^2 + |g_1(z)|^2 + |g_2(z)|^2$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$.
Then u_3 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}^m$. But u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$ if and only if u_3 is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$.
By lemma 1, we have $m = 1$ and $n \leq 2$.

Now $u(0, .)$ is convex on $\mathbb{C}$ and $u_3(0, .)$ is strictly sh on $\mathbb{C}$. In fact $(|A_1\varphi - g_1(0)|^2 + |A_2\varphi - g_2(0)|^2)$ is convex on $\mathbb{C}$ and $((|A_1|^2 + |A_2|^2)|\varphi|^2 + |g_1(0)|^2 + |g_2(0)|^2)$ is strictly sh on $\mathbb{C}$. Then there exists $c \in \mathbb{C}$ such that $|\varphi + c|^2$ is convex on $\mathbb{C}$ and $|\varphi|^2$ is strictly sh on $\mathbb{C}$. Which yields $|\varphi + c|^2$ is convex and strictly sh on $\mathbb{C}$.
By Abidi [2], using the holomorphic differential equation $k''(k + c) = \gamma(k')^2$ $(k : \mathbb{C} \to \mathbb{C}$ be a holomorphic function $, \gamma, c \in \mathbb{C})$, we have
$\varphi(w) = aw + b$, for all $w \in \mathbb{C}$, where $a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C}$, or
$\varphi(w) = e^{(a_1 w + b_1)} - c$, for all $w \in \mathbb{C}$, with $a_1 \in \mathbb{C}\backslash\{0\}$ and $b_1 \in \mathbb{C}$.
The rest of the proof is now obvious.

Theorem 5. *The following conditions are equivalent*
(I) v *is convex and strictly psh on* $\mathbb{C}^n \times \mathbb{C}^m$;
(II) $m = 1$, $n \in \{1, 2\}$, $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$, *there exists* $c \in \mathbb{C}$ *such that* $|\varphi + c|^2$ *is convex and strictly sh on* $\mathbb{C}$ *and we have the following 2 cases.*
Case 1. For all $w \in \mathbb{C}$, $\varphi(w) = aw + b$, $(a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C})$.
Then we have the representation

$$\begin{cases} g_1(z) = \overline{A_1}(< z/\lambda > +\mu) + A_2\varphi_1(z) \\ g_2(z) = \overline{A_2}(< z/\lambda > +\mu) - A_1\varphi_1(z) \end{cases}$$

for each $z \in \mathbb{C}^n$, *where* $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$, $\varphi_1 : \mathbb{C}^n \to \mathbb{C}$ *is analytic,* $|\varphi_1|^2$ *is convex on* $\mathbb{C}^n$, *such that*
$(n = 1, \lambda \neq 0)$, *or* $(n = 1, \lambda = 0, \frac{\partial \varphi_1}{\partial z}(z) \neq 0$, for every $z \in \mathbb{C})$, *or*
$(n = 2,$ and $(\lambda, (\frac{\partial \varphi_1}{\partial z_1}(z), \frac{\partial \varphi_1}{\partial z_2}(z)))$ is a basis of the complex vector space $\mathbb{C}^2$,
for any $z = (z_1, z_2) \in \mathbb{C}^2)$.
Case 2. For each $w \in \mathbb{C}$, $\varphi(w) = e^{(aw+b)} - c$, $(a \in \mathbb{C}\backslash\{0\}$ and $b \in \mathbb{C})$.
Then $n = 1$ *and we have the representation*

$$\begin{cases} g_1(z) = -\overline{A_1}\overline{c} + A_2\psi_1(z) \\ g_2(z) = -\overline{A_2}\overline{c} - A_1\psi_1(z) \end{cases}$$

for every $z \in \mathbb{C}$, *where* $\psi_1 : \mathbb{C} \to \mathbb{C}$ *is analytic,* $|\psi_1|^2$ *is convex and strictly sh on* $\mathbb{C}$.

Moreover, we can consider the function v_2 for a study. According to lemma 1, we obtain several holomorphic representations of g_1 and g_2 from the assumptions v and v_1 are convex on $\mathbb{C}^n \times \mathbb{C}^m$ and $v_2 = (v + v_1)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$.

3. Some study in the theory of convex and strictly psh functions

3.1. The analysis of strictly convex functions

Put $u(z, w) = |\varphi_1(w) - g_1(z)|^2 + |\varphi_2(w) - g_2(z)|^2$, $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be four holomorphic functions, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$.
Recall that, for two holomorphic functions $\varphi : \mathbb{C}^m \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}$, if we denote $\psi(z, w) = |\varphi(w) - g(z)|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$. ψ is not strictly convex at each point of $\mathbb{C}^n \times \mathbb{C}^m$ (this is the case of one absolute value of a holomorphic function). But, if we consider the sum of two absolute values of holomorphic functions, there exists several cases where ψ_1 is strictly convex on $\mathbb{C}^2$. For example

$$\psi_1(z, w) = |f_1(w) - k_1(z)|^2 + |f_2(w) - k_2(z)|^2$$

for $(z, w) \in \mathbb{C}^2$ and $f_1(w) = w$, $f_2(w) = 2w + 1$, $k_1(z) = 2z$, $k_2(z) = 0$.
Before the two above technical remarks, we pose the following question.
Question. Characterize all the holomorphic functions φ_1, φ_2, g_1, g_2 such that u is strictly convex on $\mathbb{C}^n \times \mathbb{C}^m$ (we prove that $n = m = 1$).

Remark 1. Let $F_1(z) = z^2$, $F_2(z) = -z^2$, $F_3(z) = z$, $K_1(w) = K_2(w) = K_3(w) = w$, $(z, w) \in \mathbb{C}^2$. $F_1, F_2, F_3, K_1, K_2, K_3$ are holomorphic functions on $\mathbb{C}$. Put $u(z, w) = |K_1(w) - F_1(z)|^2 + |K_2(w) - F_2(z)|^2 + |K_3(w) - F_3(z)|^2$. Observe that u is strictly convex on $\mathbb{C}^2$, but F_1 and F_2 are not affine functions.

We begin by

Lemma 2. *Let* $f_1, f_2 : \mathbb{C}^N \to \mathbb{C}$ *be two holomorphic functions,* $N \geq 1$. *Put* $v = |f_1|^2 + |f_2|^2$. *We have*
If v *is strictly psh on* $\mathbb{C}^N$, *then* $N \leq 2$.

Using the holomorphic differential equation $k''(k+c)=\gamma(k')^2$, for $k:\mathbb{C}\to\mathbb{C}$ be a holomorphic function and $(\gamma,c)\in\mathbb{C}^2$, we have

Lemma 3. *Let $g_1,g_2:\mathbb{C}^n\to\mathbb{C}$ and $\varphi_2:\mathbb{C}^m\to\mathbb{C}$ be three holomorphic functions and $a\in\mathbb{C}$.*
Put $u(z,w)=|g_1(z)-a|^2+|\varphi_2(w)-g_2(z)|^2$, for $(z,w)\in\mathbb{C}^n\times\mathbb{C}^m$.
Then u is strictly convex on $\mathbb{C}^n\times\mathbb{C}^m$ if and only if $n=m=1$, g_1 is affine nonconstant, g_2 is affine and φ_2 is affine nonconstant on $\mathbb{C}$.

Proof. Assume that u is strictly convex on $\mathbb{C}^n\times\mathbb{C}^m$. By lemma 2, it follows that $n=m=1$. We have

$$|\varphi_2''(w)(\overline{\varphi_2}(w)-\overline{g_2}(z))|<|\varphi_2'(w)|^2$$

for each $w\in\mathbb{C}$ and for every fixed $z\in\mathbb{C}$.
Put $\psi_2(w)=|\varphi_2(w)-g_2(z)|^2$, for $w\in\mathbb{C}$. By Abidi [2], for each fixed $z\in\mathbb{C}$, the function ψ_2 is strictly convex in $\mathbb{C}$. Then φ_2 is affine nonconstant on $\mathbb{C}$, (see [2], [3]).
Now we have the inequality

$$|g_2''(z)(\overline{g_2}(z)-\overline{\varphi_2}(w))+g_1''(z)\overline{g_1}(z)|<|g_1'(z)|^2+|g_2'(z)|^2$$

for each $(z,w)\in\mathbb{C}^2$. Therefore the function $F(w)=\overline{g_2''}(z)\varphi_2(w)$ is holomorphic and bounded on $\mathbb{C}$, for every fixed $z\in\mathbb{C}$. Therefore F is constant on $\mathbb{C}$, for each fixed $z\in\mathbb{C}$.
Since φ_2 is affine nonconstant, it follows that $g_2''=0$ on $\mathbb{C}$. Then g_2 is affine on $\mathbb{C}$.
Now write $\varphi_2(w)=A_2w+B_2$, $g_2(z)=a_2z+b_2$, $A_2\in\mathbb{C}\backslash\{0\}$, $B_2,a_2,b_2\in\mathbb{C}$. Let $T(z,w)=(z,w+\frac{a_2}{A_2}z+\frac{b_2}{A_2})$.
Thus T is an affine holomorphic transformation and bijective on $\mathbb{C}^2$. Then $u_1=u\,\mathrm{o}\,T$ is strictly convex on $\mathbb{C}^2$ and $u\,\mathrm{o}\,T(z,w)=|g_1(z)-a|^2+|\varphi_2(w)|^2=u_1(z,w)$.
Consequently, g_1 is affine nonconstant on $\mathbb{C}$.
The converse is obvious and the proof is complete.

Now let $\psi_1,\psi_2,f_1,f_2,k:\mathbb{C}\to\mathbb{C}$ be holomorphic functions and $\gamma,c\in\mathbb{C}$. Using the holomorphic differential equation $k''(k+c)=\gamma(\underline{k'})^2$ and the two partial differential equations $\psi_1''(w)\overline{f_1'}(z)+\psi_2''(w)\overline{f_2'}(z)=0$, $f_1''(z)\overline{\psi_1'}(w)+f_2''(z)\overline{\psi_2'}(w)=0$ on $\mathbb{C}^2$, we prove

Theorem 6. *Let $\varphi_1,\varphi_2:\mathbb{C}^m\to\mathbb{C}$ and $g_1,g_2:\mathbb{C}^n\to\mathbb{C}$ be four holomorphic functions. Put $u(z,w)=|\varphi_1(w)-g_1(z)|^2+|\varphi_2(w)-g_2(z)|^2$, $(z,w)\in\mathbb{C}^n\times\mathbb{C}^m$.*
The following assertions are equivalent
(I) u is strictly convex on $\mathbb{C}^n\times\mathbb{C}^m$;
(II) $n=m=1$, g_1,g_2, φ_1,φ_2 are affine functions on $\mathbb{C}$ and satisfying the condition $(g_1'\varphi_2'-g_2'\varphi_1')\neq 0$.

Proof. We have $n=m=1$, because u is strictly psh on $\mathbb{C}^n\times\mathbb{C}^m$. Since u is strictly convex on $\mathbb{C}^n\times\mathbb{C}^m$, then the function $u(z,.)$ is strictly convex on $\mathbb{C}$, for each $z\in\mathbb{C}$. Therefore,

$$|\varphi_1''(w)(\overline{\varphi_1}(w)-\overline{g_1}(z))+\varphi_2''(w)(\overline{\varphi_2}(w)-\overline{g_2}(z))|<|\varphi_1'(w)|^2+|\varphi_2'(w)|^2$$

for each $w \in \mathbb{C}^m$ and for every fixed $z \in \mathbb{C}^n$. Thus, for every fixed $w \in \mathbb{C}$, the holomorphic function on the variable z, defined by $F(z) = (g_1(z)\overline{\varphi_1''}(w) + g_2(z)\overline{\varphi_2''}(w))$ is bounded on $\mathbb{C}$.

By Liouville theorem, F is constant on $\mathbb{C}$. Thus $(g_1'(z)\overline{\varphi_1''}(w) + g_2'(z)\overline{\varphi_2''}(w)) = 0$, for every $z, w \in \mathbb{C}$.
We discuss the cases $\varphi_1'' \neq 0$ or $\varphi_2'' \neq 0$ on $\mathbb{C}$. (Also we have $(\varphi_1'(w)\overline{g_1''}(z) + \varphi_2'(w)\overline{g_2''}(z)) = 0$ on $\mathbb{C}^2$).
Assume that $\varphi_1'' \neq 0$ and $\varphi_2'' \neq 0$. Therefore

$$\frac{\varphi_1''(w)}{\varphi_2''(w)} = -\frac{\overline{g_2'}(z)}{\overline{g_1'}(z)} = R, \ \ R \in \mathbb{C}.$$

Thus, $\varphi_1''(w) = R\varphi_2''(w)$ and $g_2'(z) = -\overline{R}g_1'(z)$, for each $z, w \in \mathbb{C}$. It follows that $\varphi_1(w) = R\varphi_2(w) + aw + b$ and $g_2(z) = -\overline{R}g_1(z) + \lambda$, $a, b, \lambda \in \mathbb{C}$.
The function F_1 is strictly convex on $\mathbb{C}^2$, where

$$F_1(z, w) = |R\varphi_2(w) + aw + b - g_1(z)|^2 + |\varphi_2(w) + \overline{R}g_1(z) - \lambda|^2.$$

This proves $|g_1 + \xi_1|^2$ is strictly convex on $\mathbb{C}$, where $\xi_1 \in \mathbb{C}$.
By the holomorphic differential equation $k''(k + c) = \gamma(k')^2$, ($k : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and $c, \gamma \in \mathbb{C}$), we have g_1 is affine nonconstant on $\mathbb{C}$. Therefore, $|g_1 - \varphi_1|^2 + |\overline{R}g_1 - (\lambda - \varphi_2)|^2$ is strictly convex on $\mathbb{C}^2$.
By theorem 2, φ_1 and φ_2 are affine functions. A contradiction.
Consequently, $\varphi_1'' = 0$, or $\varphi_2'' = 0$ on $\mathbb{C}$.
Assume that $\varphi_1'' \neq 0$ and $\varphi_2'' = 0$ on $\mathbb{C}$. Therefore $\varphi_1''\overline{g_1'} = 0$ on $\mathbb{C}$. Thus $g_1' = 0$ on $\mathbb{C}$ and then g_1 is constant on $\mathbb{C}$. We have $|\varphi_1 - g_1(0)|^2 + |\varphi_2 - g_2|^2$ is strictly convex on $\mathbb{C}^2$. By lemma 3, we have φ_1 and g_2 are affine nonconstant, φ_2 is affine on $\mathbb{C}$. Therefore φ_1 is affine nonconstant on $\mathbb{C}$. A contradiction.
Consequently, φ_1 and φ_2 are affine functions on $\mathbb{C}$.
Now since the function $u(., w)$ is strictly convex on $\mathbb{C}$ (for each fixed $w \in \mathbb{C}$), then g_1, g_2, φ_1 and φ_2 satisfy the partial differential equation $g_1''\overline{\varphi_1'} + g_2''\overline{\varphi_2'} = 0$ in $\mathbb{C}^2$.
Using the last above partial differential equation, we prove that g_1 and g_2 are affine functions on $\mathbb{C}$. Note that if φ_1 and g_1 are constant functions, then $|g_2 - \varphi_2|^2$ is strictly convex on $\mathbb{C}^2$. This is impossible.
Therefore, we have
(φ_1 or g_1 is non constant) and (φ_2 or g_2 is non constant).
Analogously,
(g_1 or g_2 is non constant) and (φ_1 or φ_2 is non constant).
Since now u is strictly convex on $\mathbb{C}^2$, then

$$|\varphi_1'(w)\beta - g_1'(z)\alpha|^2 + |\varphi_2'(w)\beta - g_2'(z)\alpha|^2 > 0$$

for each $(z, w) \in \mathbb{C}^2$ and $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0\}$. Therefore, $(g_1'\varphi_2' - g_2'\varphi_1') \neq 0$.

3.2. The analysis of convex and strictly psh functions

Let $\psi_1, \psi_2, f_1, f_2, k : \mathbb{C} \to \mathbb{C}$ be holomorphic functions and $\gamma, c \in \mathbb{C}$. In the sequel, using the holomorphic differential equation $k''(k+c) = \gamma(k')^2$ and the two partial differential equations $\psi_1''(w)\overline{f_1'}(z) + \psi_2''(w)\overline{f_2'}(z) = 0$, $f_1''(z)\overline{\psi_1'}(w) + f_2''(z)\overline{\psi_2'}(w) = 0$ on $\mathbb{C}^2$, we have

Theorem 7. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ and $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ be four holomorphic functions. Put $u(z,w) = |\varphi_1(w) - g_1(z)|^2 + |\varphi_2(w) - g_2(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$. The following conditions are equivalent*
(I) u is convex and strictly psh on $\mathbb{C}^n \times \mathbb{C}^m$;
(II) $n = m = 1$, $\varphi_1''\overline{g_1'} + \varphi_2''\overline{g_2'} = 0$ and $g_1''\overline{\varphi_1'} + g_2''\overline{\varphi_2'} = 0$ on $\mathbb{C}^2$,

(φ_1 or φ_2 is nonconstant) and (g_1 or g_2 is nonconstant) and we have the following cases.
Case 1. The functions φ_1 and φ_2 satisfies $\varphi_1'' \neq 0$ and $\varphi_2'' \neq 0$.
Assume that $g_1' \neq 0$.
If $g_1'' = 0$, then $g_2'' = 0$ on $\mathbb{C}$ (therefore g_1 and g_2 are affine functions with g_1 or g_2 is non constant. In this case, by theorem 2 or theorem 3, we can find φ_1 and φ_2 by their holomorphic expressions).
If $g_1'' \neq 0$. Thus $g_2'' \neq 0$. Since $u(z,.)$ is convex on $\mathbb{C}$ (for z fixed), then $\varphi_2 = c\varphi_1 + \xi_0$, $c, \xi_0 \in \mathbb{C}$.
$u = |\varphi_1 - g_1|^2 + |c\varphi_1 + \xi_0 - g_2|^2$, on $\mathbb{C}^2$.
Assume that $g_2' \neq 0$.
We have an analogous situation to the above case.
Case 2. The function φ_1 is not affine and the function φ_2 is affine on $\mathbb{C}$.
Then g_1 is constant on $\mathbb{C}$, $|\varphi_1 - g_1(0)|^2$ and $|g_2 - \varphi_2(0)|^2$ are convex functions and $|\varphi_1' g_2'| > 0$ on $\mathbb{C}^2$, or
g_2 is affine nonconstant and $|\varphi_1' g_2'| > 0$ on $\mathbb{C}^2$.
We can study also the case $\varphi_1'' = 0$ and $\varphi_2'' \neq 0$.
Case 3. The functions φ_1 and φ_2 are affine on $\mathbb{C}$.
The discussion is similar to cases 1, 2 and theorem 3.

Proof. (I) implies (II). By lemma 2, we have $2 \leq n+m \leq 2$. Then $n = m = 1$. Since u is convex and of class C^2 on $\mathbb{C}^2$, we have the inequality

$$|\frac{\partial^2 u}{\partial w^2}\beta^2 + \frac{\partial^2 u}{\partial z^2}\alpha^2 + \frac{\partial^2 u}{\partial z \partial w}\alpha\beta| \leq \frac{\partial^2 u}{\partial w \partial \overline{w}}|\beta|^2 + \frac{\partial^2 u}{\partial z \partial \overline{z}}|\alpha|^2 + 2\mathrm{Re}(\frac{\partial^2 u}{\partial \overline{z} \partial w}\overline{\alpha}\beta)$$

on $\mathbb{C}^2$. It follows that

$$|[\varphi_1''(\overline{\varphi_1} - \overline{g_1}) + \varphi_2''(\overline{\varphi_2} - \overline{g_2})]\beta^2 + [g_1''(\overline{g_1} - \overline{\varphi_1}) + g_2''(\overline{g_2} - \overline{\varphi_2})]\alpha^2| \leq |\varphi_1'\beta - g_1'\alpha|^2 + |\varphi_2'\beta - g_2'\alpha|^2$$

for each $(\alpha, \beta) \in \mathbb{C}^2$. If $\alpha = 0$ and $\beta \neq 0$, then

$$|\varphi_1''(\overline{\varphi_1} - \overline{g_1}) + \varphi_2''(\overline{\varphi_2} - \overline{g_2})| \leq |\varphi_1'|^2 + |\varphi_2'|^2$$

on $\mathbb{C}^2$. Now let $\psi(z) = g_1(z)\overline{\varphi_1''}(w) + g_2(z)\overline{\varphi_2''}(w) - \varphi_1(w)\overline{\varphi_1''}(w) - \varphi_2(w)\overline{\varphi_2''}(w)$, for $z \in \mathbb{C}$, (w is fixed on $\mathbb{C}$). ψ is holomorphic on $\mathbb{C}$ and $\psi(z)| \leq |\varphi_1'(w)|^2 + |\varphi_2''(w)|^2$, for every $z \in \mathbb{C}$, (w fixed). Thus ψ is constant on $\mathbb{C}$. Consequently, $\psi'(z) = 0$, for each $z \in \mathbb{C}$. Therefore

$$g_1'(z)\overline{\varphi_1''}(w) + g_2'(z)\overline{\varphi_2''}(w) = 0$$

for each $z, w \in \mathbb{C}$.
Now if $\alpha \neq 0$ and $\beta = 0$. We obtain $\varphi_1'(w)\overline{g_1''}(z) + \varphi_2'(w)\overline{g_2''}(z) = 0$, for every $(z, w) \in \mathbb{C}^2$.
For the rest of the proof we use theorem 1, theorem2, theorem 3 and the proof of theorem 7.

Remark 2. Using the above technical methods, the following three partial differential equations

$$k''(k+c) = \gamma(k')^2,$$
$$\psi_1''(w)\overline{f_1'}(z) + \psi_2''(w)\overline{f_2'}(z) = 0 \text{ on } \mathbb{C}^2,$$
$$f_1''(z)\overline{\psi_1'}(w) + f_2''(z)\overline{\psi_2'}(w) = 0 \text{ on } \mathbb{C}^2,$$

where ($\psi_1, \psi_2, f_1, f_2, k : \mathbb{C} \to \mathbb{C}$ are holomorphic functions and $\gamma, c \in \mathbb{C}$), we can solve the analogous problem when u is convex on $\mathbb{C}^n \times \mathbb{C}^m$ and $u = |\varphi_1 - g_1|^2 + |\varphi_2 - g_2|^2$; $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ are four holomorphic functions with the conditions (φ_1 or φ_2 is nonconstant) and (g_1 or g_2 is nonconstant).

3.3. Essential properties in function theory

In the sequel, we give technical tools for the study of the following families of functions consisting of: convex and not strictly psh functions on any not empty Euclidean open ball subset of $\mathbb{C}^n \times \mathbb{C}$; convex and strictly sh functions but not strictly psh on each Euclidean open ball; convex and $n-$ strictly sh functions but not strictly psh on every open ball,... . We have

Theorem 8. *Let $u : D \to \mathbb{R}$ be a function of class C^2, D is a domain of $\mathbb{C}^n, n \geq 1$. The following conditions are equivalent*
(I) u is not strictly psh on each not empty Euclidean open ball subset of D;
(II) u is not strictly psh at each point of D.

Example. Let $v(z, w) = |w^N - g_1(z)|^2 + |w^N - g_2(z)|^2$, $n, N \in \mathbb{N}$, $n, N \geq 2$, $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. v is convex and not strictly psh at each point of $\mathbb{C}^n \times \mathbb{C}$, if for example $g_2(z) = -g_1(z)$, for each $z \in \mathbb{C}^n$ and $|g_1|^2$ is convex on $\mathbb{C}^n$.

Remark 3. (R1). Let $u_1(z, w) = |w - z|^2$, $u_2(z, w) = |w - 2z|^2$, $(z, w) \in \mathbb{C}^2$.
u_1 and u_2 are C^∞ and not strictly psh functions at each point of $\mathbb{C}^2$. But $u = (u_1 + u_2)$ is strictly psh on $\mathbb{C}^2$.
(R2). Put $v(z) = \| z \|^4$, $z = (z_1, ..., z_n) \in \mathbb{C}^n$. v is psh on $\mathbb{C}^n$ and strictly psh on $\mathbb{C}^n \backslash \{0\}$. Therefore v is strictly psh almost everywhere on $\mathbb{C}^n$. But v is not strictly psh on $\mathbb{C}^n$.

Example. Let $u = (u_1 + u_2)$, $v = (v_1 + v_2)$, where

$$u_1(z, w) = |w - f_1(z)|^2 + |w - f_2(z)|^2,$$

$$u_2(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2,$$
$$v_1(z,w) = |w - \overline{f_1}(z)|^2 + |w - \overline{f_2}(z)|^2,$$
$$v_2(z,w) = |w - \overline{g_1}(z)|^2 + |w - \overline{g_2}(z)|^2,$$
$$f_1(z) = -f_2(z) = (z - z^2),\ g_1(z) = -g_2(z) = (z + z^2),\ \text{for } (z,w) \in \mathbb{C}^2.$$

f_1, f_2, g_1, g_2 are holomorphic functions on $\mathbb{C}$. We have u and v are strictly convex functions on $\mathbb{C}^2$. But u_1, u_2, v_1, v_2 are not convex functions on $\mathbb{C}^2$.

Example. Let $N \in \mathbb{N}$, $N \geq 2$ and $A \in \mathbb{R}_+$, $A \geq 2$ such that ψ is convex on $\mathbb{C}$, $\psi(z) = A|z|^2 + |z^N - 1|^2$, for $z \in \mathbb{C}$. Put $u = (u_1 + u_2)$, where $u_1(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2$, $u_2(z,w) = |w - \overline{g_1}(z)|^2 + |w - \overline{g_2}(z)|^2$,
$g_1(z) = Az + (z^N - 1)$, $g_2(z) = Az - (z^N - 1)$, for $(z,w) \in \mathbb{C}^2$.
Note that g_1 and g_2 are holomorphic functions on $\mathbb{C}$. We have u_1 is not strictly psh and not convex on $\mathbb{C}^2$. u_2 is strictly psh and not convex on $\mathbb{C}^2$. But u is convex and strictly psh on $\mathbb{C}^2$.
We have

Proposition 1. *Let $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ be two holomorphic functions. Put $u(z,w) = |w - g_1(z)|^4 + |w - g_2(z)|^4$, $v(z,w) = |w - \overline{g_1}(z)|^4 + |w - \overline{g_2}(z)|^4$, for $(z,w) \in \mathbb{C}^2$. We have u is not strictly psh on $\mathbb{C}^2$, for each tuple of holomorphic functions g_1 and g_2. But there exists several cases where v is strictly psh on $\mathbb{C}^2$.*

Proof. u and v are functions of class C^∞ on $\mathbb{C}^2$. The hermitian Levi form of u is $L(u)(z,w)(\alpha,\beta) = 4|w - g_1(z)|^2|\beta - g_1'(z)\alpha|^2 + 4|w - g_2(z)|^2|\beta - g_2'(z)\alpha|^2$, for $(z,w) \in \mathbb{C}^2$, $(\alpha,\beta) \in \mathbb{C}^2$.
Let $z_0 \in \mathbb{C}$. Put $w_0 = g_1(z_0)$. Let $\beta = g_2'(z_0)\alpha$, for $\alpha \in \mathbb{C}\backslash\{0\}$.
Then $L(u)(z_0,w_0)(\alpha, g_2'(z_0)\alpha) = 0$ and $\alpha \neq 0$.
The hermitian Levi form of v is
$L(v)(z,w)(\alpha,\beta) = (2|g_1'(z)|^2|w - \overline{g_1}(z)|^2 + 2|g_2'(z)|^2|w - \overline{g_2}(z)|^2)|\alpha|^2 + (2|w - \overline{g_1}(z)|^2 + 2|w - \overline{g_2}(z)|^2)|\beta|^2 + 2|g_1'(z)(w - \overline{g_1}(z))\alpha - (\overline{w} - g_1(z))\beta|^2 + 2|g_2'(z)(w - \overline{g_2}(z))\alpha - (\overline{w} - g_2(z))\beta|^2$, for $(z,w), (\alpha,\beta) \in \mathbb{C}^2$. Now choose $|g_1'| > 0$, $|g_2'| > 0$ and $|g_1 - g_2| > 0$ on $\mathbb{C}$. Let $(z,w) \in \mathbb{C}^2$. We discuss the following three cases $(\alpha \neq 0, \beta = 0)$, $(\alpha = 0, \beta \neq 0))$ and $(\alpha \neq 0, \beta \neq 0)$, we obtain $L(v)(z,w)(\alpha,\beta) > 0$ if $(\alpha,\beta) \in \mathbb{C}^2\backslash\{0\}$.
Then v is strictly psh on $\mathbb{C}^2$.
Let $\psi(z,w) = |w - \psi_1(z)|^2 + |w - \psi_2(z)|^2 + |w - \psi_3(z)|^2$,
$\varphi(z,w) = |w - \overline{\psi_1}(z)|^2 + |w - \overline{\psi_2}(z)|^2 + |w - \overline{\psi_3}(z)|^2$, for $(z,w) \in \mathbb{C}^2$, where $\psi_1, \psi_2, \psi_3 : \mathbb{C} \to \mathbb{C}$ are three holomorphic functions. Recall that if ψ is strictly psh on $\mathbb{C}^2$, then φ is strictly psh on $\mathbb{C}^2$. But we have

Proposition 2. *There exists three holomorphic functions $g_1, g_2, g_3 : \mathbb{C} \to \mathbb{C}$ such that if we define $u(z,w) = |w - g_1(z)|^4 + |w - g_2(z)|^4 + |w - g_3(z)|^4$ and $v(z,w) = |w - \overline{g_1}(z)|^4 + |w - \overline{g_2}(z)|^4 + |w - \overline{g_3}(z)|^4$, for $(z,w) \in \mathbb{C}^2$. We have u is convex on $\mathbb{C}^2$ and strictly psh on a neighborhood of $(0,i)$. But v is not strictly psh at $(0,i)$, while v is convex on $\mathbb{C}^2$.*

Example. Let $g_1(z) = z - i$, $g_2(z) = 2z - i$, $g_3(z) = 3z - i$, $z \in \mathbb{C}$. g_1, g_2 and g_3 are holomorphic functions on $\mathbb{C}$.

$z_0 = 0, w_0 = i$. Put $u(z,w) = |w - g_1(z)|^4 + |w - g_2(z)|^4 + |w - g_3(z)|^4$,
$v(z,w) = |w - \overline{g_1}(z)|^4 + |w - \overline{g_2}(z)|^4 + |w - \overline{g_3}(z)|^4$, for $(z,w) \in \mathbb{C}^2$.
Then u and v are functions of class C^∞ and convex on $\mathbb{C}^2$.
Let $\psi(z,w) = |w - g_1(z)|^4$, $(z,w) \in \mathbb{C}^2$. ψ is a C^∞ function on $\mathbb{C}^2$ and the hermitian Levi form of ψ is

$$L(\psi)(z,w)(\alpha,\beta) = 4|w - g_1(z)|^2|\beta - g_1'(z)\alpha|^2, \quad (\alpha,\beta) \in \mathbb{C}^2.$$

Denote by $L(u)(z,w)(\alpha,\beta)$ the hermitian Levi form of u at (z,w) and (α,β). Then $L(u)(z_0,w_0)(\alpha,\beta) = 16|\beta-\alpha|^2+16|\beta-2\alpha|^2+16|\beta-3\alpha|^2 = 0$ implies that $\alpha = \beta = 0$.
Thus $L(u)(z_0,w_0)(\alpha,\beta) > 0$, for each $(\alpha,\beta) \in \mathbb{C}^2\backslash\{0\}$.
Let $S = \{(\alpha,\beta) \in \mathbb{C}^2 \ / \ |\alpha|^2 + |\beta|^2 = 1\}$. Thus $\{(z_0,w_0)\} \times S = K$ is a compact on $\mathbb{C}^2 \times \mathbb{C}^2$.
The function F, defined by

$$F(z,w)(\alpha,\beta) = \frac{\partial^2 u}{\partial z \partial \overline{z}}(z,w)|\alpha|^2 + \frac{\partial^2 u}{\partial w \partial \overline{w}}(z,w)|\beta|^2 + 2\mathrm{Re}(\frac{\partial^2 u}{\partial \overline{z} \partial w}(z,w)\overline{\alpha}\beta),$$

is continuous on $\mathbb{C}^2 \times \mathbb{C}^2$.
Since $F > 0$ on K, then $F > 0$ on $B((z_0,w_0),r) \times S$, where $r > 0$. Therefore u is strictly psh on a neighborhood of $(0,i)$ and convex on $\mathbb{C}^2$.
The hermitian Levi form of the C^∞ function θ on $\mathbb{C}^2$ is

$$\begin{aligned} L(\theta)(z,w)(\alpha,\beta) &= 2|g_1'(z)(w - \overline{g_1}(z))\alpha - (\overline{w} - g_1(z))\beta|^2 + 2|g_1'(z)(\overline{w} - g_1(z))\alpha|^2 \\ &+ 2|w - \overline{g_1}(z)|^2|\beta|^2, \end{aligned}$$

for $(z,w), (\alpha,\beta) \in \mathbb{C}^2$, where $\theta(z,w) = |w - \overline{g_1}(z)|^4$.
Observe that we have $w_0 - \overline{g_1}(z_0) = w_0 - \overline{g_2}(z_0) = w_0 - \overline{g_3}(z_0) = 0$. Therefore $L(v)(z_0,w_0)(\alpha,\beta) = 0$, for each $(\alpha,\beta) \in \mathbb{C}^2$.

We have the following technical remark.

Remark 4. Let $f_1, ..., f_N : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions, $n, N, k \in \mathbb{N}\backslash\{0\}$, $k \geq 2$. Put

$$u(z,w) = |w - f_1(z)|^{2k} + ... + |w - f_N(z)|^{2k},$$
$$v(z,w) = |w - \overline{f_1}(z)|^{2k} + ... + |w - \overline{f_N}(z)|^{2k},$$
$$u_1(z,w) = |w - f_1(z)|^2 + ... + |w - f_N(z)|^2,$$
$$v_1(z,w) = |w - \overline{f_1}(z)|^2 + ... + |w - \overline{f_N}(z)|^2,$$
$$\varphi = (u + v) \text{ and } \varphi_1 = (u_1 + v_1).$$

If u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, we can not deduce that v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
If φ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, we can not conclude that u (or v) is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
But we have the technical properties.
(I) If u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, then u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
(II) v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ implies that v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.

(III) If φ_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, then v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
(IV) If φ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, then φ_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
(V) $(u+u_1)$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$, implies that u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
For example for the proof of the above property (I), since
$u(z,w) = |(w-f_1(z))^k|^2 + ... + |(w-f_N(z))^k|^2$, then u is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Therefore the hermitian Levi form of u is

$$\begin{aligned} L(u)(z,w)(\alpha,\beta) &= |w-f_1(z)|^{2k-2}|\beta - \sum_{j=1}^{n} \frac{\partial f_1}{\partial z_j}(z)\alpha_j|^2 + ... \\ &+ |w-f_N(z)|^{2k-2}|\beta - \sum_{j=1}^{n} \frac{\partial f_N}{\partial z_j}(z)\alpha_j|^2 \end{aligned}$$

for $z=(z_1,...,z_n) \in \mathbb{C}^n$, $w \in \mathbb{C}$, $\alpha = (\alpha_1,...,\alpha_n) \in \mathbb{C}^n$, $\beta \in \mathbb{C}$.
Now u_1 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. The hermitian Levi form of u_1 is

$$L(u_1)(z,w)(\alpha,\beta) = |\beta - \sum_{j=1}^{n} \frac{\partial f_1}{\partial z_j}(z)\alpha_j|^2 + ... + |\beta - \sum_{j=1}^{n} \frac{\partial f_N}{\partial z_j}(z)\alpha_j|^2.$$

Let $(z,w),(\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C}$. Observe that $L(u)(z,w)(\alpha,\beta) > 0$ implies that $L(u_1)(z,w)(\alpha,\beta) > 0$, because the absolute value $|\beta - \sum_{j=1}^{n} \frac{\partial f_s}{\partial z_j}(z)\alpha_j|^2 \geq 0$, for each $s \in \{1,...,N\}$.
The technical properties (II), (III), (IV) and (V) can be be proved similarly.
Observe that for $\psi : \mathbb{C}^n \to \mathbb{R}_+$, if ψ^2 is convex on $\mathbb{C}^n$, then ψ^4 is convex on $\mathbb{C}^n$. The converse, for instance, is in general not true. But in the sequel, using the holomorphic differential equation, $k''(k+c) = \gamma(k')^2$ ($k : \mathbb{C} \to \mathbb{C}$ be holomorphic and $c, \gamma \in \mathbb{C}$), we have

Theorem 9. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. Put $u(z,w) = |w-g_1(z)|^2 + |w-g_2(z)|^2$, $v(z,w) = |w-g_1(z)|^4 + |w-g_2(z)|^4$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We have*
(I) Assume that v is convex on $\mathbb{C}^n \times \mathbb{C}$, then u is convex on $\mathbb{C}^n \times \mathbb{C}$.
(II) Suppose that u is convex on $\mathbb{C}^n \times \mathbb{C}$, we can not conclude that v is convex on $\mathbb{C}^n \times \mathbb{C}$.

Proof. (I). Note that u and v are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$.
Assume that $n=1$. We have

$$|\frac{\partial^2 v}{\partial z^2}(z,w)\alpha^2 + \frac{\partial^2 v}{\partial w^2}\beta^2 + 2\frac{\partial^2 v}{\partial z \partial w}\alpha\beta| \leq L(v)(z,w)(\alpha,\beta)$$

for each $(z,w),(\alpha,\beta) \in \mathbb{C}^2$, where

$$L(v)(z,w)(\alpha,\beta) = \frac{\partial^2 v}{\partial z \partial \overline{z}}(z,w)|\alpha|^2 + \frac{\partial^2 v}{\partial w \partial \overline{w}}|\beta|^2 + 2\mathrm{Re}(\frac{\partial^2 v}{\partial z \partial \overline{w}}\alpha\overline{\beta}).$$

We obtain the inequality
(E): $|[-2g_1''(z)w + 2g_1(z)g_1''(z) + 2(g_1'(z))^2](\overline{w}^2 - 2\overline{g_1}(z)\overline{w} + \overline{g_1}^2(z))\alpha^2 +$
$[-2g_2''(z)w + 2g_2(z)g_2''(z) + 2(g_2'(z))^2](\overline{w}^2 - 2\overline{g_2}(z)\overline{w} + \overline{g_2}^2(z))\alpha^2 + 2(\overline{w} - \overline{g_1}(z))^2\beta^2 +$
$2(\overline{w} - \overline{g_2}(z))^2\beta^2 - 2g_1'(z)(\overline{w} - \overline{g_1}(z))^2\alpha\beta - 2g_2'(z)(\overline{w} - \overline{g_2}(z))^2\alpha\beta| \leq$
$|2w\beta - 2\beta g_1(z) - 2wg_1'(z)\alpha + 2g_1'(z)g_1(z)\alpha|^2 + |2w\beta - 2\beta g_2(z) - 2wg_2'(z)\alpha + 2g_2'(z)g_2(z)\alpha|^2$,
for each (z, w), $(\alpha, \beta) \in \mathbb{C}^2$.
If $\beta = 0$ and $w \in \mathbb{R}$, the coefficient of w^3 is equal to 0. Therefore $(g_1''(z) + g_2''(z)) = 0$, for every $z \in \mathbb{C}$.
Now we divide the left hand side of the inequality (E) by $|\overline{w}|^2 > 0$ (for $w \in \mathbb{C}\backslash\{0\}$) and the right hand side of (E) by $|w|^2$ (observe that $|\overline{w}|^2 = |w|^2$), and letting $|w|$ go to $(+\infty)$, we obtain

$$|(4g_1''(z)\overline{g_1}(z) + 4g_2''(z)\overline{g_2}(z) + 2(g_1'(z))^2 + 2(g_2'(z))^2)\alpha^2 + 4\beta^2 - 4(g_1'(z) + g_2'(z))\alpha\beta| \leq |2\beta - 2g_1'(z)\alpha|^2 + |2\beta - 2g_2'(z)\alpha|^2.$$

Put $\beta = g_1'(z)\alpha$. Then

$$|4g_1''(z)\overline{g_1}(z) + 4g_2''(z)\overline{g_2}(z) + 2(g_1'(z) - g_2'(z))^2| \leq 4|g_1'(z) - g_2'(z)|^2.$$

Thus

$$|g_1''(z)(\overline{g_1}(z) - \overline{g_2}(z))|^2 \leq 6|g_1'(z) - g_2'(z)|^2$$

for each $z \in \mathbb{C}$. Now also we prove that

$$|g_2''(z)(\overline{g_1}(z) - \overline{g_2}(z))| \leq 6|g_1'(z) - g_2'(z)|^2$$

for every $z \in \mathbb{C}$. Using the triangle inequality, we have then

$$|g_1''(z)(g_1(z) - g_2(z)) - g_2''(z)(g_1(z) - g_2(z))| \leq 12|g_1'(z) - g_2'(z)|^2$$

for each $z \in \mathbb{C}$.
Therefore the function $(g_1 - g_2)$ satisfies

$$|(g_1''(z) - g_2''(z))(g_1(z) - g_2(z))| \leq 12|g_1'(z) - g_2'(z)|^2$$

for every $z \in \mathbb{C}$. Therefore the function $|g_1 - g_2|^2$ is convex on $\mathbb{C}$, by Abidi [2], (we can see [3]).
Since $(g_1 + g_2)$ is affine on $\mathbb{C}$, thus $g_1(z) = (az + b) + \varphi(z)$, $g_2(z) = (az + b) - \varphi(z)$, for each $z \in \mathbb{C}$, where $\varphi : \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that $|\varphi|$ is convex on $\mathbb{C}$. Therefore u is convex on $\mathbb{C}^2$.
In the sequel, we can prove that g_1 and g_2 are affine functions on $\mathbb{C}$ (see proposition 3).
Assume that $n \geq 2$. Actually by the above case, it is easy to prove that g_1 and g_2 are affine functions on every complex line $L \subset \mathbb{C}^n$. Therefore, g_1 and g_2 are affine functions on $\mathbb{C}^n$.
(II). Assume that $n = 1$. Put $g_1(z) = z^2$, $g_2(z) = -z^2$, for $z \in \mathbb{C}$. Then

$$u(z, w) = |w - g_1(z)|^2 + |w - g_2(z)|^2 = 2|w|^2 + 2|z|^2, \ (z, w) \in \mathbb{C}^2.$$

Thus u is convex on $\mathbb{C}^2$. But v is not convex on $\mathbb{C}^2$, because $v(z,1) = |1-z^2|^4 + |1+z^2|^4 = 2\psi(z)$, for each $z \in \mathbb{C}$. Observe that ψ is not convex in a neighborhood of $\frac{1}{2}$.

Proposition 3. *Let* $u(z,w) = |w+ < z/a > +b+\varphi(z)|^4 + |w+ < z/a > +b-\varphi(z)|^4$, $a \in \mathbb{C}^n$, $b \in \mathbb{C}$, $\varphi : \mathbb{C}^n \to \mathbb{C}$ *be holomorphic not affine,* $|\varphi|$ *is convex on* $\mathbb{C}^n$.
Then the function u *is not convex on* $\mathbb{C}^n \times \mathbb{C}$.

Proof. Define $v(z,w) = |w+\varphi(z)|^4 + |w-\varphi(z)|^4$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Observe that v is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if u is convex on $\mathbb{C}^n \times \mathbb{C}$.
Suppose that $n = 1$. Since φ is not affine and $|\varphi|$ is convex on $\mathbb{C}$, then by Abidi [3], we have the holomorphic representations
$\varphi(z) = (a_1 z + b_1)^k$, for each $z \in \mathbb{C}$, where $a_1 \in \mathbb{C}\backslash\{0\}$, $b_1 \in \mathbb{C}$, $k \in \mathbb{N}$, $k \geq 2$, or
$\varphi(z) = e^{(a_2 z + b_2)}$, for every $z \in \mathbb{C}$, with $a_2 \in \mathbb{C}\backslash\{0\}$ and $b_2 \in \mathbb{C}$.
Now for the study of the convexity of the function v, by an affine change of variable, we can assume that $\varphi(z) = z^k$, for any $z \in \mathbb{C}$, or $\varphi(z) = e^z$, for each $z \in \mathbb{C}$.
(I) Assume that $\varphi(z) = z^k$, $k \in \mathbb{N}$, $k \geq 2$.
If $k = 2$. We can see the above proof and we have the function $F = v(.,1)$ is not convex on $\mathbb{C}$.
Now suppose that $k \geq 3$.
Define $\psi(z) = v(z,1)$, for $z \in \mathbb{C}$. Then ψ is a function of class C^∞ on $\mathbb{C}$.
If ψ is convex on $\mathbb{C}$, then

$$\left|\frac{\partial^2 \psi}{\partial z^2}(z)\right| \leq \frac{\partial^2 \psi}{\partial z \partial \overline{z}}(z)$$

for each $z \in \mathbb{C}$.

$$\begin{aligned}\frac{\partial^2 \psi}{\partial z^2}(z) &= [2k^2 z^{2k-2} + 2k(k-1)z^{k-2}(1+z^k)](1+\overline{z}^k)^2 \\ &+ [2k^2 z^{2k-2} + 2k(k-1)z^{k-2}(z^k-1)](\overline{z}^k-1)^2. \\ \frac{\partial^2 \psi}{\partial z \partial \overline{z}}(z) &= 4k^2|z^{2k-2}|1+z^k|^2 + 4k^2|z^k-1|^2|z|^{2k-2}.\end{aligned}$$

For $z_0 = 1$, $\frac{\partial^2 \psi}{\partial z^2}(1) = 4(6k^2-4k) \geq 0$ and $\frac{\partial^2 \psi}{\partial z \partial \overline{z}}(1) = 16k^2$. Then $\frac{\partial^2 \psi}{\partial z^2}(1) = |\frac{\partial^2 \psi}{\partial z^2}(1)| \leq \frac{\partial^2 \psi}{\partial z \partial \overline{z}}(1)$. Therefore $6k^2 - 4k \leq 4k^2$ and $k \geq 3$. This is a contradiction.
(II) Assume that $\varphi(z) = e^z$, for $z \in \mathbb{C}$.
Let $\psi(z) = v(z,2)$, $z \in \mathbb{C}$. ψ is a function of class C^∞ on $\mathbb{C}$.

$$\begin{aligned}\frac{\partial^2 \psi}{\partial z^2}(z) &= 2(2e^z + 2e^{2z})(2+e^{\overline{z}})^2 + 2(2e^{2z} - 2e^z)(e^{\overline{z}}-2)^2. \\ \frac{\partial^2 \psi}{\partial z \partial \overline{z}}(z) &= 4e^{(z+\overline{z})}(e^z+2)(e^{\overline{z}}+2) + 4e^{(z+\overline{z})}(e^z-2)(e^{\overline{z}}-2).\end{aligned}$$

$\frac{\partial^2 \psi}{\partial z^2}(0) = 72$ and $\frac{\partial^2 \psi}{\partial z \partial \overline{z}}(0) = 40$. Therefore $|\frac{\partial^2 \psi}{\partial z^2}(0)| > \frac{\partial^2 \psi}{\partial z \partial \overline{z}}(0)$. Then ψ is not convex on $\mathbb{C}$. Consequently, v is not convex on $\mathbb{C}^2$.
Comparing the preceding theorem and proposition 3, we observe that the exponent 2 is

special in our considerations. For instance, let $u_k(z,w) = |w-f_1(z)|^{2k}+|w-f_2(z)|^{2k}$, $k \in \mathbb{N}\backslash\{0\}$, $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions and $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We can prove that u_k is convex on $\mathbb{C}^n \times \mathbb{C}$ implies that u_1 is convex on $\mathbb{C}^n \times \mathbb{C}$ if $(k \geq 2)$, but the converse is not true.
Let $v_\delta(z,w) = |A_1w - f_1(z)|^\delta + |A_2w - f_2(z)|^\delta$, $\delta \in [1,+\infty[$ and $(A_1, A_2) \in \mathbb{C}^2\backslash\{0\}$. Observe that the study of the convexity of the function v_δ is based on two additional cases.
Moreover, observe that by the above technical proof, we have

Theorem 10. *Let $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions. Define $u(z,w) = |w - f_1(z)|^4 + |w - f_2(z)|^4$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We have u is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if f_1 and f_2 are affine functions on $\mathbb{C}^n$.*

Proof. We can see the proof of theorem 9 and proposition 3.

Remark 5. Let $f_1(z) = z^N$, $f_2(z) = -z^N$, $f_3(z) = iz^N$ and $f_4(z) = -iz^N$, $N \in \mathbb{N}\backslash\{0,1\}$, for $z \in \mathbb{C}$.
Put $u(z,w) = |w - f_1(z)|^4 + |w - f_2(z)|^4 + |w - f_3(z)|^4 + |w - f_4(z)|^4$, $(z,w) \in \mathbb{C}^2$. u is convex on $\mathbb{C}^2$, because $u(z,w) = c(|w|^2 + |z^N|^2)^2$, where $c \in \mathbb{R}$, $c > 0$. But f_1, f_2, f_3 and f_4 are not affine functions.

We have the following.

Question 1. Let $F_1, F_2, F_3 : \mathbb{C}^n \to \mathbb{C}$ be holomorphic functions. Put $\psi_1(z) = (|F_1(z)|^4 + |F_2(z)|^4)$, $\psi_2(z) = (|F_1(z)|^4 + |F_2(z)|^4 + |F_3(z)|^4)$, $z \in \mathbb{C}^n$.
(I) Is it true that ψ_1 is convex on $\mathbb{C}^n$ implies that F_1 and F_2 are affine functions on $\mathbb{C}^n$?
(II) Assume that ψ_2 is convex on $\mathbb{C}^n$. Is it true that F_1, F_2 and F_3 are affine functions on $\mathbb{C}^n$?
The number of holomorphic functions is it fundamental in the above two situations?

We have

Proposition 4. *Let $k \in \mathbb{N}\backslash\{0,1\}$ and $\varphi : \mathbb{C}^n \to \mathbb{C}$ be holomorphic. Define $v(z,w) = |w+ < z/a > +b+\varphi(z)|^{2k}+|w+ < z/a > +b-\varphi(z)|^{2k}$, $a \in \mathbb{C}^n, b \in \mathbb{C}$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Assume that φ is not affine and $|\varphi|$ is convex on $\mathbb{C}^n$. Then v is not convex on $\mathbb{C}^n \times \mathbb{C}$.*

Proof. Obviously follows from the proof of proposition 3. Observe that, using the holomorphic differential equation cited above, we have the additional result.

Theorem 11. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions and $k \in \mathbb{N}\backslash\{0,1\}$. Put $u(z,w) = |w - g_1(z)|^{2k} + |w - g_2(z)|^{2k}$ and $v(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.*
(I) Assume that u is convex on $\mathbb{C}^n \times \mathbb{C}$. Then v is convex on $\mathbb{C}^n \times \mathbb{C}$.
(II) Suppose that v is convex on $\mathbb{C}^n \times \mathbb{C}$. We can not conclude that u is convex on $\mathbb{C}^n \times \mathbb{C}$. But we have
(III) u is convex on $\mathbb{C}^n \times \mathbb{C}$ if and only if g_1 and g_2 are affine functions.

Extension of the results. Let $\psi_\delta = |w - f_1(z)|^\delta + |w - f_2(z)|^\delta$, $\delta \in [1,+\infty[$, $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be two holomorphic functions and $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We observe without any assumption on $\delta \in [1,+\infty[$, for instance , for the study of the convexity

of the function ψ_δ, the proof is organized in two separately cases.
Case 1. $\delta = 2$. (In this case, we obtain several solutions not affine functions).
Case 2. $\delta \in [1, +\infty[\backslash\{2\}$.
In general we have the following two remarks (R1) and (R2).
(R1). Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Put $\varphi_\delta(z, w) = |w - f(z)|^\delta$, $\delta \in [1, +\infty[$ and $(z, w) \in \mathbb{C}^2$. We have φ_δ is convex on $\mathbb{C}^2$ if and only if f is affine (and in particular f is a function of class C^∞ on $\mathbb{C}$).
(Let $N \in \mathbb{N}\backslash\{0\}$, $2N \geq \delta$ and put $G(z, w) = |w - f(z)|^{2N}$, $(z, w) \in \mathbb{C}^2$. Suppose that φ_δ is convex on $\mathbb{C}^2$. Consequently, G is psh on $\mathbb{C}^2$. By Abidi [1], it follows that f is harmonic on $\mathbb{C}$. Now let $T : \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}-$ linear bijective transformation. Consider $M(z, w) = (T(z), w)$, for $(z, w) \in \mathbb{C}^2$. Note that M is $\mathbb{R}-$ linear and a bijective transformation on $\mathbb{C}^2$. Therefore $G \,\mathrm{o}\, M$ is convex on $\mathbb{C}^2$ and consequently, $G \,\mathrm{o}\, M$ is psh on $\mathbb{C}^2$. Since $G \,\mathrm{o}\, M(z, w) = |w - f \,\mathrm{o}\, T(z)|$, for $(z, w) \in \mathbb{C}^2$. Then $f \,\mathrm{o}\, T$ is harmonic on $\mathbb{C}$, for any $\mathbb{R}-$ linear transformation T. Then f is affine on $\mathbb{C}$).
But if we define $F_\delta(z, w) = |w - f_1(z)|^\delta + |w - g_1(z)|^\delta$, where

$$f_1(z) = \begin{cases} 1 \text{ if } \mathrm{Re}(z) \geq 0 \\ -1 \text{ if } \mathrm{Re}(z) < 0 \end{cases}$$

and

$$g_1(z) = \begin{cases} -1 \text{ if } \mathrm{Re}(z) \geq 0 \\ 1 \text{ if } \mathrm{Re}(z) < 0 \end{cases}$$

for $(z, w) \in \mathbb{C}^2$. Then we have
$F_\delta(z, w) = |w - 1|^\delta + |w + 1|^\delta$ and consequently, the function F_δ is convex on $\mathbb{C}^2$, for each $\delta \geq 1$. But f_1 and g_1 are noncontinuous functions at any point of $\mathbb{C}$. Moreover, we have
(R2). There exists two continuous functions $f, g : \mathbb{C} \to \mathbb{C}$, with $K_\delta(z, w) = |w - f(z)|^\delta + |w - g(z)|^\delta$, $(z, w) \in \mathbb{C}^2$, K_δ is convex on $\mathbb{C}^2$ (for each $\delta \geq 1$), but f and g are not functions of class C^∞ on $\mathbb{C}$.
Example. Let $f(z) = |x|$, $g(z) = -|x|$, $z = (x + iy) \in \mathbb{C}$, $x = \mathrm{Re}(z)$.

Question 2. Let $\psi_1, ..., \psi_N : \mathbb{C}^n \to \mathbb{C}$ be analytic functions, $N, k \in \mathbb{N}$, $k \geq 2$. Define

$$\psi(z, w) = |w - \psi_1(z)|^{2k} + ... + |w - \psi_N(z)|^{2k},\ (z, w) \in \mathbb{C}^n \times \mathbb{C}.$$

Assume that $N \leq 2k - 1$ and ψ is convex on $\mathbb{C}^n \times \mathbb{C}$. Characterize $\psi_1, ..., \psi_N$ by their analytic expressions.

Question 3. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : \mathbb{C}^m \to \mathbb{C}$ and $g_1, g_2, g_3, g_4 : \mathbb{C}^n \to \mathbb{C}$ be 8 holomorphic functions. Put $u = (u_1 + u_2)$, where $u_1(z, w) = |\varphi_1(w) - g_1(z)|^4 + |\varphi_2(w) - g_2(z)|^4$, $u_2(z, w) = |\varphi_3(w) - g_3(z)|^4 + |\varphi_4(w) - g_4(z)|^4$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
Characterize $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, g_1, g_2, g_3, g_4 by their expressions such that u_1 and u_2 are convex functions on $\mathbb{C}^n \times \mathbb{C}$ and u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$.
In the sequel, for instance, observe that there exists a great differences between the exponent 2 and the exponent 4 (or $2k, k \in \mathbb{N}\backslash\{0, 1\}$) in real convexity.

We have

Lemma 4. *(I) There exists* $\psi_1, \psi_2 : \mathbb{C}^n \to \mathbb{C}$ *two holomorphic functions such that* $|\psi_1|^2$ *and* $|\psi_2|^2$ *are not convex functions, while* $u = (|\psi_1|^2 + |\psi_2|^2)$ *is convex on* $\mathbb{C}^n$*, but* $v = (|\psi_1|^4 + |\psi_2|^4)$ *is not convex on* $\mathbb{C}^n$ *(respectively* $(|\psi_1|^{2k} + |\psi_2|^{2k})$ *is not convex on* $\mathbb{C}^n$ *for each* $k \in \mathbb{N}\backslash\{0,1\}$*).*
(II) There exists $\varphi_1, \varphi_2 : \mathbb{C}^n \to \mathbb{C}$ *holomorphic functions, with* $|\varphi_1|^2$ *is convex and* $|\varphi_2|^2$ *is not convex on* $\mathbb{C}^n$*,* $(|\varphi_1|^2 + |\varphi_2|^2)$ *is convex on* $\mathbb{C}^n$*, but* $(|\varphi_1|^{2k} + |\varphi_2|^{2k})$ *is not convex on* $\mathbb{C}^n$*, for each* $k \in \mathbb{N}\backslash\{0,1\}$*.*
(Example. $\varphi_1(z) = 2z$*,* $\varphi_2(z) = z^2 - 1$*,* $z \in \mathbb{C}$*).*

We introduce this lemma because it yields the following questions.

Question 4. Let $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ be analytic functions and $\delta \in [1, +\infty[$. Put $u = (|f_1|^\delta + |f_2|^\delta)$. Suppose that u is convex on $\mathbb{C}^n$ and $\delta \neq 2$. Is it true that $|f_1|$ and $|f_2|$ are convex functions on $\mathbb{C}^n$?

Question 5. Let $n, m \in \mathbb{N}\backslash\{0\}$. Find all the holomorphic functions $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$, $\varphi_1, \varphi_2 : \mathbb{C}^m \to \mathbb{C}$, such that ψ is convex on $\mathbb{C}^n \times \mathbb{C}^m$, where $\psi(z,w) = |\varphi_1(w) - f_1(z)|^\delta + |\varphi_2(w) - f_2(z)|^\delta$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.

4. Some study of a particular case and algebraic method

Theorem 12. *Let* $A_1, A_2, A_3, A_4, A_5 \in \mathbb{C}\backslash\{0\}$*. Consider* $g_1, g_2, g_3, g_4, g_5 : \mathbb{C}^n \to \mathbb{C}$ *be five holomorphic functions. Define* $u_1(z,w) = |A_1 w - g_1(z)|^2 +$
$|A_2 w - g_2(z)|^2$*,* $v_1(z,w) = |A_3 w - g_3(z)|^2 + |A_4 w - g_4(z)|^2$*,* $u(z,w) = u_1(z,w) + v_1(z,w) + |A_5 w - g_5(z)|^2$*,* $(z,w) \in \mathbb{C}^n \times \mathbb{C}$*.*
The following conditions are equivalent
(I) u_1 *and* v_1 *are convex functions on* $\mathbb{C}^n \times \mathbb{C}$ *and* u *is (convex and strictly psh) on* $\mathbb{C}^n \times \mathbb{C}$*;*
(II) $n \in \{1,2,3,4\}$ *and we have*

$$\begin{cases} g_1(z) = A_1(< z/a > +b) + \overline{A_2}\varphi(z) \\ g_2(z) = A_2(< z/a > +b) - \overline{A_1}\varphi(z) \end{cases}$$

$$\begin{cases} g_3(z) = A_3(< z/c > +d) + \overline{A_4}\psi(z) \\ g_4(z) = A_4(< z/c > +d) - \overline{A_3}\psi(z) \end{cases}$$

and $g_5(z) = (< z/\lambda > +\mu)$*, (for all* $z \in \mathbb{C}^n$*, where* $a, c, \lambda \in \mathbb{C}^n$*,* $b, d, \mu \in \mathbb{C}$*,* $\varphi, \psi : \mathbb{C}^n \to \mathbb{C}$ *are 2 holomorphic functions,* $|\varphi|$ *and* $|\psi|$ *are convex functions on* $\mathbb{C}^n$*) with the following 4 cases.*
(1) $n = 4$*. We have* $(a - c, a - \lambda, (\overline{\frac{\partial\varphi}{\partial z_1}}(z), \overline{\frac{\partial\varphi}{\partial z_2}}(z), \overline{\frac{\partial\varphi}{\partial z_3}}(z), \overline{\frac{\partial\varphi}{\partial z_4}}(z)),$
$(\overline{\frac{\partial\psi}{\partial z_1}}(z), \overline{\frac{\partial\psi}{\partial z_2}}(z), \overline{\frac{\partial\psi}{\partial z_3}}(z), \overline{\frac{\partial\psi}{\partial z_4}}(z)))$ *is a basis of the complex vector space* $\mathbb{C}^4$*, for all* $z \in \mathbb{C}^4$*.*
(2) $n = 3$*. Then we have for all* $z \in \mathbb{C}^3$*,* $z = (z_1, z_2, z_3)$*,*
$(a - c, a - \lambda, (\overline{\frac{\partial\varphi}{\partial z_1}}(z), \overline{\frac{\partial\varphi}{\partial z_2}}(z), \overline{\frac{\partial\varphi}{\partial z_3}}(z)))$*, or* $(a - c, a - \lambda, (\overline{\frac{\partial\psi}{\partial z_1}}(z), \overline{\frac{\partial\psi}{\partial z_2}}(z), \overline{\frac{\partial\psi}{\partial z_3}}(z)))$*, or*

$(a-c,(\overline{\frac{\partial\varphi}{\partial z_1}(z)},\overline{\frac{\partial\varphi}{\partial z_2}(z)},\overline{\frac{\partial\varphi}{\partial z_3}(z)}),(\overline{\frac{\partial\psi}{\partial z_1}(z)},\overline{\frac{\partial\psi}{\partial z_2}(z)},\overline{\frac{\partial\psi}{\partial z_3}(z)}))$, or $(a-\lambda,(\overline{\frac{\partial\varphi}{\partial z_1}(z)},\overline{\frac{\partial\varphi}{\partial z_2}(z)},\overline{\frac{\partial\varphi}{\partial z_3}(z)}),(\overline{\frac{\partial\psi}{\partial z_1}(z)},\overline{\frac{\partial\psi}{\partial z_2}(z)},\overline{\frac{\partial\psi}{\partial z_3}(z)}))$ is a basis of the complex vector space $\mathbb{C}^3$.

(3) $n=2$. Then for each $z=(z_1,z_2)\in\mathbb{C}^2$, the quantity $(\overline{a-c},\overline{a-\lambda})$, or $(a-c,(\overline{\frac{\partial\varphi}{\partial z_1}(z)},\overline{\frac{\partial\varphi}{\partial z_2}(z)}))$, or $(a-c,(\overline{\frac{\partial\psi}{\partial z_1}(z)},\overline{\frac{\partial\psi}{\partial z_2}(z)}))$, or $(a-\lambda,(\overline{\frac{\partial\varphi}{\partial z_1}(z)},\overline{\frac{\partial\varphi}{\partial z_2}(z)}))$, or $(a-\lambda,(\overline{\frac{\partial\psi}{\partial z_1}(z)},\overline{\frac{\partial\psi}{\partial z_2}(z)}))$, or $((\overline{\frac{\partial\varphi}{\partial z_1}(z)},\overline{\frac{\partial\varphi}{\partial z_2}(z)}),(\overline{\frac{\partial\psi}{\partial z_1}(z)},\overline{\frac{\partial\psi}{\partial z_2}(z)}))$ is a basis of the complex vector space $\mathbb{C}^2$.

(4) $n=1$. Then we have for all $z\in\mathbb{C}$, $(a-c)\neq 0$, or $(a-\lambda)\neq 0$, or $(\frac{\partial\varphi}{\partial z}(z)\neq 0)$, or $(\frac{\partial\psi}{\partial z}(z)\neq 0)$.

Proof. (I) implies (II). Since u_1 is convex on $\mathbb{C}^n\times\mathbb{C}$, then

$$\begin{cases} g_1(z)=A_1(<z/a>+b)+\overline{A_2}\varphi(z)\\ g_2(z)=A_2(<z/a>+b)-\overline{A_1}\varphi(z)\end{cases}$$

(for each $z\in\mathbb{C}^n$, where $a\in\mathbb{C}^n$, $b\in\mathbb{C}$, $\varphi:\mathbb{C}^n\to\mathbb{C}$ is analytic, $|\varphi|$ is convex on $\mathbb{C}^n$). u_2 is convex on $\mathbb{C}^n\times\mathbb{C}$, then

$$\begin{cases} g_3(z)=A_3(<z/c>+d)+\overline{A_4}\psi(z)\\ g_4(z)=A_4(<z/c>+d)-\overline{A_3}\psi(z)\end{cases}$$

(for every $z\in\mathbb{C}^n$, with $c\in\mathbb{C}^n$, $d\in\mathbb{C}$, $\psi:\mathbb{C}^n\to\mathbb{C}$ is analytic and $|\psi|$ is convex on $\mathbb{C}^n$).

Note that u is a function of class C^∞ on $\mathbb{C}^n\times\mathbb{C}$. Now since u is convex on $\mathbb{C}^n\times\mathbb{C}$, then if we put $z=(z_1,...,z_n)\in\mathbb{C}^n$, $w=z_{n+1}$, $\alpha=(\alpha_1,...,\alpha_n)\in\mathbb{C}^n$, $\beta=\alpha_{n+1}\in\mathbb{C}$, we have

$$\left|\sum_{j,k=1}^{n+1}\frac{\partial^2 u}{\partial z_j\partial z_k}(z)\alpha_j\alpha_k\right|\leq\sum_{j,k=1}^{n+1}\frac{\partial^2 u}{\partial z_j\partial\overline{z_k}}(z)\alpha_j\overline{\alpha}_k,\ \forall(z,w)\in\mathbb{C}^n\times\mathbb{C},\ \forall(\alpha,\beta)\in\mathbb{C}^n\times\mathbb{C}.$$

It follows that g_5 is an affine function on $\mathbb{C}^n$. Therefore $g_5(z)=(<z/\lambda>+\mu)$, for each $z\in\mathbb{C}^n$, where $\lambda\in\mathbb{C}^n$ and $\mu\in\mathbb{C}$. Then

$$\begin{aligned} u(z,w) &= (|A_1|^2+|A_2|^2)(|w-<z/a>-b|^2+|\varphi(z)|^2)\\ &+ (|A_3|^2+|A_4|^2)(|w-<z/c>-d|^2+|\psi(z)|^2)\\ &+ |A_5w-<z/\lambda>-\mu|^2,\ (z,w)\in\mathbb{C}^n\times\mathbb{C}.\end{aligned}$$

Define

$$\begin{aligned} v(z,w) &= |w-<z/a>-b|^2+|\varphi(z)|^2+|w-<z/c>-d|^2\\ &+ |\psi(z)|^2+|A_5w-<z/\lambda>-\mu|^2,\ (z,w)\in\mathbb{C}^n\times\mathbb{C}.\end{aligned}$$

Then v is a function of class C^∞ on $\mathbb{C}^n\times\mathbb{C}$ and we have (u is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ if and only if v is strictly psh on $\mathbb{C}^n\times\mathbb{C}$). Therefore by lemma 1, $n+1\leq 5$. Consequently, $n\in\{1,2,3,4\}$.

Now let $T(z,w) = (z, w+ < z/a >)$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. T is a $\mathbb{C}-$ linear bijective transformation on $\mathbb{C}^n \times \mathbb{C}$.
Let $v_2(z,w) = v \circ T(z,w) = |w-b|^2 + |\varphi(z)|^2 + |w+ < z/a - c > -d|^2 + |\psi(z)|^2 + |A_5 w+ < z/a - \lambda > -\mu|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.
Therefore v_2 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. We have v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. The Levi hermitian form of v_2 is
$L(v_2)(z,w)(\alpha,\beta) = |\beta|^2 + |\sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j}(z)\alpha_j|^2 + |\beta+ < \alpha/a - c > |^2 +$
$|\sum_{j=1}^{n} \frac{\partial \psi}{\partial z_j}(z)\alpha_j|^2 + |A_5\beta+ < \alpha/a - \lambda > |^2$, $(z,w) = ((z_1,...,z_n),w) \in \mathbb{C}^n \times \mathbb{C}$, $(\alpha,\beta) = ((\alpha_1,...,\alpha_n),\beta) \in \mathbb{C}^n \times \mathbb{C}$.
Now $L(v_2)(z,w)(\alpha,\beta) = 0$ if and only if $\beta = 0$ and

$$|\sum_{j=1}^{n} \frac{\partial \varphi}{\partial z_j}(z)\alpha_j|^2 + |\beta+ < \alpha/a - c > |^2 + |\sum_{j=1}^{n} \frac{\partial \psi}{\partial z_j}(z)\alpha_j|^2 + |A_5\beta+ < \alpha/a - \lambda > |^2 = 0,$$

$(z,w), (\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C}$. It follows that, if we define $u_2(z) = |\varphi(z)|^2 + | < z/a - c > -d|^2 + |\psi(z)|^2 + | < z/a - \lambda > -\mu|^2$, for $z \in \mathbb{C}^n$, then u_2 is a function of class C^∞ on $\mathbb{C}^n$.
Now Observe that v_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if u_2 is strictly psh on $\mathbb{C}^n$.
Case 1. $n = 4$. In this case observe that u is strictly psh on $\mathbb{C}^4 \times \mathbb{C}$ if and only if the quantity

$$(a-c, a-\lambda, (\overline{\frac{\partial \varphi}{\partial z_1}}(z), \overline{\frac{\partial \varphi}{\partial z_2}}(z), \overline{\frac{\partial \varphi}{\partial z_3}}(z), \overline{\frac{\partial \varphi}{\partial z_4}}(z)), (\overline{\frac{\partial \psi}{\partial z_1}}(z), \overline{\frac{\partial \psi}{\partial z_2}}(z), \overline{\frac{\partial \psi}{\partial z_3}}(z), \overline{\frac{\partial \psi}{\partial z_4}}(z)))$$

is a basis of the complex vector space $\mathbb{C}^4$, for all $z = (z_1,z_2,z_3,z_4) \in \mathbb{C}^4$.
Case 2. $n = 3$. The Levi hermitian form of u_2 is
$L(u_2)(z)(\alpha) = |\sum_{j=1}^{3} \frac{\partial \varphi}{\partial z_j}(z)\alpha_j|^2 + | < \alpha/a - c > |^2 + |\sum_{j=1}^{3} \frac{\partial \psi}{\partial z_j}(z)\alpha_j|^2 +$
$| < \alpha/a - \lambda > |^2$, for $z = (z_1,z_2,z_3) \in \mathbb{C}^3$, $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{C}^3$.
$L(u_2)(z)(\alpha) = 0$ if and only if

$$\begin{cases} < \alpha/a - c >= 0, \\ < \alpha/a - \lambda >= 0, \\ \sum_{j=1}^{3} \frac{\partial \varphi}{\partial z_j}(z)\alpha_j = 0, \text{ and} \\ \sum_{j=1}^{3} \frac{\partial \psi}{\partial z_j}(z)\alpha_j = 0. \end{cases}$$

Therefore u_2 is strictly psh on $\mathbb{C}^3$ if and only if for all $z = (z_1,z_2,z_3) \in \mathbb{C}^3$, we can choose a basis (of the complex vector space $\mathbb{C}^3$) consisting of 3 vectors from the set of vectors

$\{a-c, a-\lambda, (\overline{\frac{\partial\varphi}{\partial z_1}}(z), \overline{\frac{\partial\varphi}{\partial z_2}}(z), \overline{\frac{\partial\varphi}{\partial z_3}}(z)), (\overline{\frac{\partial\psi}{\partial z_1}}(z), \overline{\frac{\partial\psi}{\partial z_2}}(z), \overline{\frac{\partial\psi}{\partial z_3}}(z))\}$.
Case 3. $n=2$. In this case u_2 is strictly psh on $\mathbb{C}^2$ if and only if for all $z=(z_1,z_2)\in\mathbb{C}^2$, we can choose a basis (consisting by 2 vectors basis of the complex vector space $\mathbb{C}^2$) from the set $\{a-c, a-\lambda, (\overline{\frac{\partial\varphi}{\partial z_1}}(z), \overline{\frac{\partial\varphi}{\partial z_2}}(z)), (\overline{\frac{\partial\psi}{\partial z_1}}(z), \overline{\frac{\partial\psi}{\partial z_2}}(z))\}$.
Case 4. $n=1$. u_2 is strictly sh on $\mathbb{C}$ if and only if for all $z\in\mathbb{C}$, we have $(a-c)\neq 0$, or $(a-\lambda)\neq 0$, or $(\frac{\partial\varphi}{\partial z}(z)\neq 0)$, or $(\frac{\partial\psi}{\partial z}(z)\neq 0)$.
The proof is now finished.

Moreover, we have

Question 6. Let $n,m,N\in\mathbb{N}\backslash\{0\}$ and $(A_1,B_1),...,(A_N,B_N)\in\mathbb{C}^2\backslash\{(0,0)\}$. Find all the holomorphic functions $g_1,f_1,...,g_N,f_N:\mathbb{C}^n\to\mathbb{C}$ and all the holomorphic (respectively prh) nonconstant functions $k_1,...,k_N:\mathbb{C}^m\to\mathbb{C}$ such that $u_1,...,u_N$ are convex and $u=(u_1+...+u_N)$ is strictly psh on $\mathbb{C}^n\times\mathbb{C}^m$, where $u_j(z,w)=|A_jk_j(w)-f_j(z)|^2+|B_jk_j(w)-g_j(z)|^2$, for $(z,w)\in\mathbb{C}^n\times\mathbb{C}^m$, $1\leq j\leq N$. In general we prove that this question have applications in the theory of (partial differential equations and (convex and strictly psh functions) in several variables), and therefore for the resolution of certain holomorphic partial differential equations in complex analysis. Because, in the sequel, we have a relation between partial differential equations and the subject (convex and strictly psh functions) in complex analysis and geometry.

Example. Find all the holomorphic functions $f,g:\mathbb{C}\to\mathbb{C}$, such that
(a) $|f^2+f|$ and $|g^2-g|$ are convex functions on $\mathbb{C}$, and
(b) ψ is strictly psh on $\mathbb{C}^2$, where $\psi(z_1,z_2)=|f^2(z_1)+f(z_1)|^2+|g^2(z_2)-g(z_2)|^2$, for $(z_1,z_2)\in\mathbb{C}^2$.
In this case we solve the holomorphic differential equation
$(f^2+f)''(f^2+f)=\gamma(2ff'+f')^2$, where $\gamma\in\{\frac{s-1}{s}, 1\ /\ s\in\mathbb{N}\backslash\{0\}\},...$

Example. Let $N\geq 2$. Find all the holomorphic functions $f_1,...,f_N:\mathbb{C}^n\to\mathbb{C}$, such that v is convex and strictly psh on $\mathbb{C}^n\times\mathbb{C}$. We can see the problem v is convex and v_1 is strictly psh on $\mathbb{C}^n\times\mathbb{C}$ (in this case we apply lemma 1). Where

$$v(z,w)=|w-f_1(z)|^2+...+|w-f_N(z)|^2,$$

$$v_1(z,w)=|w-\overline{f_1}(z)|^2+...+|w-\overline{f_N}(z)|^2,$$

for $(z,w)\in\mathbb{C}^n\times\mathbb{C}$. In this situation, we solve several holomorphic partial differential equations which characterize the complex structure strictly psh. Finally, we choose the solution which gives the convexity of v (or conversely).

Question 7. Let $n,m,k\in\mathbb{N}\backslash\{0\}$. Find all the holomorphic functions $g_1,g_2,g_3,g_4:\mathbb{C}^n\to\mathbb{C}$ and all the holomorphic functions $\varphi_1,\varphi_2,\varphi_3,\varphi_4:\mathbb{C}^m\to\mathbb{C}$ such that v_1 and v_2 are convex and $v=(v_1+v_2)$ is strictly convex on $\mathbb{C}^n\times\mathbb{C}^m$, where

$$v_1(z,w)=|\varphi_1(w)-g_1(z)|^{2k}+|\varphi_2(w)-g_2(z)|^{2k},$$

$$v_2(z,w)=|\varphi_3(w)-g_3(z)|^{2k}+|\varphi_4(w)-g_4(z)|^{2k},$$

for $(z,w)\in\mathbb{C}^n\times\mathbb{C}^m$.

5. The product of several psh functions and applications

The main objective of this section is to study the behaviour of the product of several absolute values of prh functions. Note that it is well known that the product of many psh functions is not in general psh.

Example. Let $v_1(z,w) = |w-\overline{z}||w-2\overline{z}|$, for $(z,w) \in \mathbb{C}^2$. Then v_1 is not psh on $\mathbb{C}^2$. In the sequel, let D be a domain of $\mathbb{C}^n$, $n \in \mathbb{N}\backslash\{0\}$, $N \in \mathbb{N}\backslash\{0,1\}$ and $\varphi_1, ..., \varphi_N : D \to \mathbb{C}$ be holomorphic functions. Define $u(z,w) = \prod_{1\leq j\leq N} |w - \overline{\varphi_j}(z)|$, for $(z,w) \in D \times \mathbb{C}$. Find conditions should satisfy N, $\varphi_1, ..., \varphi_N$ so that u is psh on $D \times \mathbb{C}$.
Now let $f_0, ..., f_{N-1} : D \to \mathbb{C}$ be holomorphic functions. Put $v(z,w) = |w^N + \overline{f_{N-1}}(z)w^{N-1} + ... + \overline{f_1}(z)w + \overline{f_0}(z)|$, $(z,w) \in D \times \mathbb{C}$. Characterize N, $f_{N-1}, ..., f_1, f_0$ such that v is psh on $D \times \mathbb{C}$.

Proposition 5. Let $f, g : D \to \mathbb{C}$ be two functions, D is a domain of $\mathbb{C}^n$, $n \geq 1$. Put $u(z,w) = |w^2 + f(z)w + g(z)|$, $(z,w) \in D \times \mathbb{C}$. Assume that f is continuous and g of class C^2 on D. Then u is psh on $D \times \mathbb{C}$ if and only if we have one assertion of the following conditions.
(I) f is holomorphic on D and g is prh on D.
(II) f is prh and not holomorphic and $g = \frac{f^2}{4}$ on D.

Proof. Put $v = u^2$. Assume that u is psh on $D \times \mathbb{C}$. Then v is psh on $D \times \mathbb{C}$. By Abidi [2], f is pluriharmonic (prh) on D. Thus v is a function of class C^2 on $D \times \mathbb{C}$. Without loss of generality we assume that $n = 1$. Let $(z,w) \in D \times \mathbb{C}$.

$$\frac{\partial^2 v}{\partial w \partial \overline{w}}(z,w) = |2w + f(z)|^2.$$

$$\frac{\partial^2 v}{\partial \overline{z} \partial w}(z,w) = \frac{\partial f}{\partial \overline{z}}(z)((\overline{w})^2 + \overline{w}\overline{f}(z) + \overline{g}(z)) + (2w + f(z))(\overline{w}\frac{\partial \overline{f}}{\partial \overline{z}}(z) + \frac{\partial \overline{g}}{\partial \overline{z}}(z)).$$

We have

$$|\frac{\partial^2 v}{\partial \overline{z} \partial w}(z,w)|^2 \leq \frac{\partial^2 v}{\partial w \partial \overline{w}}(z,w)\frac{\partial^2 v}{\partial z \partial \overline{z}}(z,w)$$

for each $(z,w) \in \mathbb{C}^2$. Now observe that if $w = -\frac{f(z)}{2}$, then $\frac{\partial^2 v}{\partial \overline{w} \partial w}(z, -\frac{f(z)}{2}) = 0$.
It follows that $\frac{\partial^2 v}{\partial \overline{z} \partial w}(z, -\frac{f(z)}{2}) = 0 = \frac{\partial f}{\partial \overline{z}}(z)(\overline{g}(z) - \frac{\overline{f^2}(z)}{4})$, for each $z \in D$. Now since f is real analytic on D, then $\frac{\partial f}{\partial \overline{z}}(z) = 0$, for every $z \in D$, or there exists $z_0 \in D$, such that $\frac{\partial f}{\partial \overline{z}}(z_0) \neq 0$.
Case 1. For each $z \in D$, $\frac{\partial f}{\partial \overline{z}}(z) = 0$.
Then f is holomorphic on D. Since $u(z,w) = |(w + \frac{f(z)}{2})^2 - \frac{f^2(z)}{4} + g(z)|$, for $(z,w) \in D \times \mathbb{C}$, we consider $T(z,w) = (z, w - \frac{f(z)}{2})$, for $(z,w) \in D \times \mathbb{C}$. T is a biholomorphism on $D \times \mathbb{C}$. Therefore $u \circ T$ is psh on $D \times \mathbb{C}$. $u \circ T(z,w) = |w^2 - \frac{f^2(z)}{4} + g(z)|$, $(z,w) \in D \times \mathbb{C}$. By Abidi [1], the function $(\frac{f^2}{4} - g)$ is harmonic on D. Consequently, g is harmonic on D.

Case 2. There exists $z_0 \in D$ such that $\frac{\partial f}{\partial \overline{z}}(z_0) \neq 0$.
We consider $E = \{\xi \in D \ / \ \frac{\partial f}{\partial \overline{\xi}}(\xi) = 0\}$. Since $\frac{\partial f}{\partial \overline{\xi}}$ is antianalytic on D, then E is an analytic closed subset on D. Therefore, $D\backslash E$ is a domain dense on D. Now since the function $(\frac{f^2}{4} - g)$ is continuous on D and $(\frac{f^2}{4} - g) = 0$ on $D\backslash E$, then $(\frac{f^2}{4} - g) = 0$ on D.
Let us mention that, if $n \geq 2$ and $f = (f_1 + \overline{f_2})$ is not holomorphic on an open polydisc $P = P_1 \times ... \times P_n \subset D$, where $f_1, f_2 : P \to \mathbb{C}$ are holomorphic functions, $P_1, ..., P_n$ are discs on $\mathbb{C}$. Since f_2 is nonconstant on P, we assume that $|\frac{\partial f_2}{\partial z_1}| > 0$ on P.
Thus $|\frac{\partial^2 v}{\partial \overline{z_1} \partial w}|^2 \leq \frac{\partial^2 v}{\partial \overline{z_1} \partial z_1} \frac{\partial^2 v}{\partial \overline{w} \partial w}$ on P. Since $\frac{\partial^2 v}{\partial \overline{w} \partial w}(z, -\frac{\overline{f(z)}}{2}) = 0$, then $\frac{\partial^2 v}{\partial \overline{z_1} \partial w}(z, -\frac{\overline{f(z)}}{2}) = 0$, for each $z \in P$. We obtain $\frac{\partial f_2}{\partial z_1}[\frac{f^2}{4} - g] = 0$ on P. Consequently, $g = \frac{f^2}{4}$ on P.
Now since f is not holomorphic on each not empty open polydisc subset of D, it follows that $g = \frac{f^2}{4}$ on D. The proof in now complete.

Now we have

Theorem 13. *Let $f, g, k : \mathbb{C}^n \to \mathbb{C}$, $n \geq 1$.*
Define $u(z, w) = |w^3 + w^2 f(z) + wg(z) + k(z)|$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
Assume that f is continuous on $\mathbb{C}^n$ and g and k are functions of class C^2 on $\mathbb{C}^n$.
Then u is psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if we have the following two cases.
Case 1. f and g are holomorphic functions and k is prh on $\mathbb{C}^n$.
Case 2. f is prh and not holomorphic on $\mathbb{C}^n$.
Put $q(w) = 3w^2 + 2wf(z) + g(z)$, for each $w \in \mathbb{C}$ and every fixed z on $\mathbb{C}^n$.
q have an only one zero on $\mathbb{C}$, for each z fixed on $\mathbb{C}^n$, (therefore $g(z) = \frac{f^2(z)}{3}$ and $k(z) = \frac{f^3(z)}{27}$).

Proof. Put $v = u^2$. Assume that u is psh on $\mathbb{C}^n \times \mathbb{C}$. Then v is psh on $\mathbb{C}^n \times \mathbb{C}$. We can prove that f is prh on $\mathbb{C}^n$, using Abidi [2]. Therefore v is a function of class C^2 on $\mathbb{C}^n \times \mathbb{C}$.
Case 1. The function f is holomorphic on $\mathbb{C}^n$.
$w^3 + w^2 f(z) + wg(z) + k(z) = (w + \frac{f(z)}{3})^3 + w(g(z) - \frac{f^2(z)}{3}) - \frac{f^3(z)}{27} + k(z)$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$.
Since psh functions are invariant by any change by holomorphic functions, we can replace $(w + \frac{f(z)}{3})$ by w, we obtain
$w^3 + (w - \frac{f(z)}{3})(g(z) - \frac{f^2(z)}{3}) - \frac{f^3(z)}{27} + k(z) = w^3 + w(g(z) - \frac{f^2(z)}{3}) + k_1(z)$, k_1 is a function of class C^2 on $\mathbb{C}^n$.
Now using the proof described in [2], we can prove that g is prh on $\mathbb{C}^n$. Suppose that g is holomorphic on $\mathbb{C}^n$. We can prove that k is prh on $\mathbb{C}^n$. Therefore $u = |h|$, where $h : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$ is prh. Then u is psh on $\mathbb{C}^n \times \mathbb{C}$. Suppose that g is not holomorphic on $\mathbb{C}^n$. Assume that $n = 1$. We have

$$\begin{aligned}
\frac{\partial^2 v}{\partial w\partial \overline{w}}(z,w) &= |3w^2+2wf(z)+g(z)|^2.\\
\frac{\partial^2 v}{\partial w\partial \overline{z}}(z,w) &= (2w\frac{\partial f}{\partial \overline{z}}(z)+\frac{\partial g}{\partial \overline{z}}(z))(w^3+f(z)w^2+g(z)w+k(z))\\
&+ (3w^2+2wf(z)+g(z))((\overline{w})^2\frac{\partial \overline{f}}{\partial \overline{z}}(z)+\overline{w}\frac{\partial \overline{g}}{\partial \overline{z}}(z)+\frac{\partial \overline{k}}{\partial \overline{z}}(z)).
\end{aligned}$$

Since v is psh then we have the inequality

$$(E):\ |\frac{\partial^2 v}{\partial w\partial \overline{z}}(z,w)|^2 \leq \frac{\partial^2 v}{\partial w\partial \overline{w}}(z,w)\frac{\partial^2 v}{\partial z\partial \overline{z}}(z,w)$$

for each $(z,w)\in\mathbb{C}^2$.
Since g is not holomorphic on $\mathbb{C}$, then there exists $z_0\in\mathbb{C}$, such that $|\frac{\partial g}{\partial \overline{z}}|>0$ on a neighborhood of z_0.
Let $q_1(w)=w^3+w^2f(z)+wg(z)+k(z)$ and $q_2(w)=3w^2+2wf(z)+g(z)$, for $(z,w)\in\mathbb{C}^2$.
Note that q_1 and q_2 are holomorphic polynomials in the variable $w\in\mathbb{C}$, for each fixed $z\in\mathbb{C}$. Also $q_1'=q_2$. The holomorphic polynomial q_2 has two zeros denoted w_1 and $w_2\in\mathbb{C}$.
Assume that $w_1\neq w_2$. Then w_1 and w_2 are distinct zeros of the polynomial q_1 by the inequality (E). Since $q_1'=q_2$ then w_1 and w_2 are two distinct zeros of order 2 of q_1. A contradiction because $\deg(q_1)=3$. Therefore $w_1=w_2$ is a zero of q_2 of order 2. Thus w_1 is a zero of q_1 of order 3. Then we have $q_1(w)=(w-w_1)^3$, for every $w\in\mathbb{C}$. Consequently, $f=-3w_1$ and then $q_1(w)=(w+\frac{f(z)}{3})^3$, for each z in a neighborhood of z_0. Then $g(z)=\frac{f^2(z)}{3}$ and therefore g is holomorphic in a neighborhood of z_0. A contradiction. This step is impossible.
Case 2. The function f is not holomorphic on $\mathbb{C}^n$.
Assume that $n=1$. Therefore $\frac{\partial f}{\partial \overline{z}}\neq 0$. Put $q_1(w)=w^3+w^2f(z)+wg(z)+k(z)$, $q_2(w)=3w^2+2wf(z)+g(z)$, $q_3(w)=2w\frac{\partial f}{\partial \overline{z}}(z)+\frac{\partial g}{\partial \overline{z}}(z)$, for $(z,w)\in\mathbb{C}^2$. Note that q_1, q_2 and q_3 are holomorphic polynomials in the variable $w\in\mathbb{C}$, for every fixed $z\in\mathbb{C}$. We have $q_1'=q_2$. Let $z_0\in\mathbb{C}$, such that $\frac{\partial f}{\partial \overline{z}}(z)\neq 0$, for every $z\in V_0$, where V_0 is an Euclidean open disc in $\mathbb{C}$, $z_0\in V_0$. Now q_2 have two zeros $w_0(z)$ and $w_1(z)\in\mathbb{C}$. Suppose that $w_0(z)=w_1(z)$. From the inequality (E), w_0 is a zero of q_1. Since $q_2'=q_1$, then w_0 is a zero of q_1 of order 3. Therefore $q_1(w)=(w-w_0)^3$. If for every $z\in V_0$, $w_0(z)=w_0=w_1(z)=w_1$, then $q_1(w)=(w-w_0(z))^3=(w+\frac{f(z)}{3})^3$, in $V_0\times\mathbb{C}$. Then $g=\frac{f^2}{3}$ and $k=\frac{f^3}{27}$. If there exists $z_1\in V_0$ such that $w_0(z_1)=a\neq w_1(z_1)=b$. The condition a and b are zeros of q_1 is impossible because $\deg(q_1)=3$. By the inequality (E), for example we have b is a zero of q_1 of order 2 and a is a zero of q_3.
Let w_2 the second zero of q_1 of order 1. Then we have the following relations between the zeros and the coefficients of the polynomial q_3, $a+b=-2\frac{f}{3}$, $ab=3g$ and $2b+w_2=-f$. Thus we have the equalities $3a+3b=4b+2w_2$, $3a=b+2w_2$ and $b^2+2bw_2=g=\frac{1}{3}ab$.

If $b \neq 0$ on a neighborhood of z_1., then $b + 2w_2 = \frac{1}{3}a = 3a$. Consequently, $a = 0$. Therefore $g = 0$ and $k \neq 0$. We have then $b = -2\frac{f}{3}$, $w_2 = \frac{f}{3}$. Thus u defined by, $u(z,w) = |w + 2\frac{f}{3}(z)|^2|w - \frac{f}{3}(z)|$, is psh on $\mathbb{C}^2$. Put $u_1(z,w) = |w + 2f(z)|^2|w - f(z)|$, for $(z,w) \in \mathbb{C}^2$. Then u_1 is psh on $\mathbb{C}^2$. But it is obvious (by theorem 14 below), that f is holomorphic on $\mathbb{C}$. A contradiction. Consequently, $b = 0$ on a neighborhood of z_1. Thus $w_2 = -f$ and $a = 0$ (because $g = 0$ on a neighborhood of z_1). Therefore $q_2(w) = 3w^2$ and observe that $f = 0$. A contradiction. Therefore, the assumption $a \neq b$ is impossible. It follows that $w_0(z) = w_1(z)$, for each $z \in V_0$. Therefore $q_1(w) = (w - w_0)^3$, for each $w \in \mathbb{C}$. We obtain $g = \frac{f^2}{3}$ and $k = \frac{f^3}{27}$.
Assume now that $n \geq 2$. Obviously we consider in this situation an analogous proof of the above theorem as well. The proof is now complete.

Recall that for each $f : D \to \mathbb{C}$, ψ is psh on $D \times \mathbb{C}$ if and only if f is pluriharmonic (prh) on D, where $\psi(z,w) = |w - f(z)|^N$, $N \in \mathbb{N}\backslash\{0\}$, D is a domain of $\mathbb{C}^n$ and $(z,w) \in D \times \mathbb{C}$. Now we prove that there exists a similar characterization of holomorphic functions. We have

Theorem 14. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be continuous. Put $u(z,w) = |w + 2f(z)|^2|w - f(z)|$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Then u is psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if f is holomorphic on $\mathbb{C}^n$.*

Proof. Assume that u is psh on $\mathbb{C}^n \times \mathbb{C}$. Since $u(z,w) = |w^3 + 3f(z)w^2 - 4f^3(z)|$, for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Then f is prh on $\mathbb{C}^n$, (see [2], page 336). In particular, f is a function of class C^∞ on $\mathbb{C}^n$. If f is holomorphic on $\mathbb{C}^n$, then u is psh on $\mathbb{C}^n \times \mathbb{C}$. Assume that f is not holomorphic on $\mathbb{C}^n$. Then f is nonconstant. Without loss of generality we suppose that $n = 1$ in all of the rest of the proof.
Case 1. The function $g = \overline{f}$ is holomorphic on $\mathbb{C}$.
Put $v = u^2$. Then $v(z,w) = |w^3 + 3\overline{g}(z)w^2 - 4\overline{g^3}(z)|^2$, $(z,w) \in \mathbb{C}^2$. Note that v is a function of class C^∞ on $\mathbb{C}^2$. We have

$$\begin{aligned}
\frac{\partial^2 v}{\partial \overline{z} \partial w}(z,w) &= 6\frac{\partial \overline{g}}{\partial \overline{z}}(z)w[\overline{w^3 + 3\overline{g}(z)w^2 - 4\overline{g^3}(z)}]. \\
\frac{\partial^2 v}{\partial \overline{w} \partial w}(z,w) &= |3w^2 + 6\overline{g}(z)w|^2, \\
\frac{\partial^2 v}{\partial \overline{z} \partial z}(z,w) &= |3\frac{\partial \overline{g}}{\partial \overline{z}}(z)w^2 - 12(\overline{g})^2(z)\frac{\partial \overline{g}}{\partial \overline{z}}(z)|^2.
\end{aligned}$$

Suppose that $\frac{\partial g}{\partial z} = 0$ on $\mathbb{C}$. Then g is constant on $\mathbb{C}$. It follows that f is constant on $\mathbb{C}$. A contradiction. Consequently, $\frac{\partial g}{\partial z} \neq 0$. Since $\frac{\partial^2 v}{\partial \overline{z} \partial z}(z, 2\overline{g}(z)) = 0$ and $|\frac{\partial^2 v}{\partial \overline{z} \partial w}(z,w)|^2 \leq \frac{\partial^2 v}{\partial \overline{z} \partial z}(z,w)\frac{\partial^2 v}{\partial \overline{w} \partial w}(z,w)$, for each $(z,w) \in \mathbb{C}^2$, then $\frac{\partial^2 v}{\partial \overline{z} \partial w}(z, 2\overline{g}(z)) = 0$, for any $z \in \mathbb{C}$. Thus $\overline{g}(z)[16(\overline{g}^3)(z)] = 0$, for all $z \in \mathbb{C}$. It follows that $g = 0$ on $\mathbb{C}$. A contradiction. Therefore this case is impossible.
Case 2. The function $g = \overline{f}$ is not holomorphic on $\mathbb{C}$.
Let $v = u^2$. Then v is a function of class C^∞ and psh on $\mathbb{C}^2$. Let $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ be two harmonic functions and $(z,w) \in \mathbb{C}^2$. Define $F(z,w) = (w - g_1(z))^2(\overline{w} - \overline{g_1}(z))^2(w - g_2(z))(\overline{w} - \overline{g_2}(z))$. Note that F is a C^∞ function on $\mathbb{C}^2$. Assume that F is psh on $\mathbb{C}^2$. We have

$\frac{\partial^2 F}{\partial \overline{w} \partial w}(z,w) = |2(w-g_2(z)) + (w-g_1(z))|^2 |w-g_1(z)|^2$.
$\frac{\partial^2 F}{\partial \overline{z} \partial w}(z,w) = -2\frac{\partial g_1}{\partial \overline{z}}(z)(\overline{w}-\overline{g_1}(z))^2(w-g_2(z))(\overline{w}-\overline{g_2}(z)) - 4\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))(w-g_2(z))(\overline{w}-\overline{g_2}(z)) - 2\frac{\partial g_2}{\partial \overline{z}}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))^2(\overline{w}-\overline{g_2}(z)) - 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))^2(w-g_2(z)) - \frac{\partial \overline{g_2}}{\partial \overline{z}}(z)(w-g_1(z))^2(\overline{w}-\overline{g_1}(z))^2 - 2\frac{\partial g_1}{\partial \overline{z}}(z)(w-g_1(z))(\overline{w}-\overline{g_2}(z))(\overline{w}-\overline{g_1}(z))^2 - 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)(\overline{w}-\overline{g_1}(z))(\overline{w}-\overline{g_2}(z))(w-g_1(z))^2$.

$\frac{\partial^2 F}{\partial \overline{z} \partial z}(z,w) = 2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z)(\overline{w}-\overline{g_1}(z))^2(w-g_2(z))(\overline{w}-\overline{g_2}(z)) + 4\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z)(\overline{w}-\overline{g_1}(z))(w-g_1(z))(w-g_2(z))(\overline{w}-\overline{g_2}(z)) + 2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial g_2}{\partial z}(w-g_1(z))(z)(\overline{w}-\overline{g_1}(z))^2(\overline{w}-\overline{g_2}(z)) + 2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))^2(w-g_2(z)) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z)(w-g_1(z))^2(w-g_2(z))(\overline{w}-\overline{g_2}(z)) + 4\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))(w-g_2(z))(\overline{w}-\overline{g_2}(z)) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial g_2}{\partial z}(z)(\overline{w}-\overline{g_1}(z))(w-g_1(z))^2(\overline{w}-\overline{g_2}(z)) +$
$2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z)(\overline{w}-\overline{g_1}(z))(w-g_1(z))^2(w-g_2(z)) + 2\frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))^2(\overline{w}-\overline{g_2}(z)) + 2\frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z)(w-g_1(z))(\overline{w}-\overline{g_1}(z))^2(\overline{w}-\overline{g_2}(z)) +$
$2\frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z)(\overline{w}-\overline{g_1}(z))(\overline{w}-\overline{g_2}(z))(w-g_1(z))^2 + \frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z)(w-g_1(z))^2(\overline{w}-\overline{g_1}(z))^2 + 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z)(w-g_1(z))(w-g_2(z))(\overline{w}-\overline{g_1}(z))^2 + 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z)(\overline{w}-\overline{g_1}(z))(w-g_1(z))^2(w-g_2(z))$.
Let $\eta > 0$. Observe that if we replace g_1 and g_2 respectively by ηg_1 and ηg_2, the new function F_1, defined by $F_1(z,w) = |w-\eta g_1(z)|^4 |w-\eta g_2(z)|^2$ for $(z,w) \in \mathbb{C}^2$, is also of class C^∞ and psh on $\mathbb{C}^2$.
Therefore if we divide by η^2 and letting η go to 0, then

$$\lim_{\eta \to 0^+} \frac{1}{\eta^2} |\frac{\partial^2 F_1}{\partial \overline{z} \partial w}(z,w)|^2 \leq \lim_{\eta \to 0^+} [\frac{1}{\eta^2} \frac{\partial^2 F_1}{\partial \overline{w} \partial w}(z,w) \frac{\partial^2 F_1}{\partial \overline{z} \partial z}(z,w)].$$

Let $N \in \mathbb{N} \backslash \{0\}$. Write $f = f_1 + \overline{f_2}$, where f_1 and f_2 are holomorphic functions on $\mathbb{C}$. Consider $T(z,w) = (z, w + Nf_1(z))$, $(z,w) \in \mathbb{C}^2$. T is a biholomorphism of $\mathbb{C}^2$. Therefore $u \operatorname{o} T$ is a function of class C^∞ and psh on $\mathbb{C}^2$.

$$u \operatorname{o} T(z,w) = |w + (N+2)f_1(z) 2\overline{f_2}(z)|^2 |w + (N-1)f_1(z) - \overline{f_2}(z)|.$$

Define $g_1 = -(N+2)f_1 + 2\overline{f_2}$ and $g_2 = -(N-1)f_1 + \overline{f_2}$ on $\mathbb{C}$.
g_1 and g_2 are harmonic functions on $\mathbb{C}$.
Thus for $w_0 = 1$ and using the above inequality and letting η go to 0
(we replace g_1 and g_2 respectively by ηg_1 and ηg_2).
We obtain
$|2\frac{\partial g_1}{\partial \overline{z}}(z) + 4\frac{\partial \overline{g_1}}{\partial \overline{z}}(z) + 2\frac{\partial g_2}{\partial \overline{z}}(z) + 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z) + \frac{\partial \overline{g_2}}{\partial \overline{z}}(z) + 2\frac{\partial g_1}{\partial \overline{z}}(z) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)|^2 \leq$

$9[2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z) + 4\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z) + 2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial g_2}{\partial z}(z) + 2\frac{\partial g_1}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z) + 4\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial g_2}{\partial z}(z) + 2\frac{\partial \overline{g_1}}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z) + 2\frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z) + 2\frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z) + \frac{\partial g_2}{\partial \overline{z}}(z)\frac{\partial \overline{g_2}}{\partial z}(z) + 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z)\frac{\partial g_1}{\partial z}(z) + \frac{\partial \overline{g_2}}{\partial \overline{z}}(z)\frac{\partial g_2}{\partial z}(z) + 2\frac{\partial \overline{g_2}}{\partial \overline{z}}(z)\frac{\partial \overline{g_1}}{\partial z}(z)]$.

Then $|4\frac{\partial \overline{f_2}}{\partial \overline{z}}(z) + 4(N+2)\frac{\partial \overline{f_1}}{\partial \overline{z}}(z) + 2\frac{\partial \overline{f_2}}{\partial \overline{z}}(z) + 3(N-1)\frac{\partial \overline{f_1}}{\partial \overline{z}}(z) + 4\frac{\partial \overline{f_2}}{\partial \overline{z}}(z) + 2(N+2)\frac{\partial \overline{f_1}}{\partial \overline{z}}(z)|^2 \leq$
$9[4(N+2)^2|\frac{\partial f_1}{\partial z}(z)|^2 + 2(N+2)(N-1)|\frac{\partial f_1}{\partial z}(z)|^2 + 2(N-1)(N+2)|\frac{\partial f_1}{\partial z}(z)|^2 + A(N,z)]$,

where $A(N,z)$ is a function defined on $\mathbb{N}\times\mathbb{C}$ and satisfy $\lim_{N\to+\infty}\frac{1}{N^2}A(N,z)=0$, for each z fixed on $\mathbb{C}$.
We divide the last above inequality by N^2 and letting N go to $+\infty$. We obtain $9\times 9|\frac{\partial f_1}{\partial z}(z)|^2 \leq 9\times 8|\frac{\partial f_1}{\partial z}(z)|^2$, for all $z\in\mathbb{C}$. Thus $\frac{\partial f_1}{\partial z}(z)=0$, for each $z\in\mathbb{C}$.
Consequently, f_1 is constant on $\mathbb{C}$. Write $f_1=c$, $c\in\mathbb{C}$. Therefore $f=c+\overline{f_2}$ on $\mathbb{C}$. It follows that $g=\overline{f}$ is holomorphic on $\mathbb{C}$. A contradiction.
Consequently, this case is impossible. Therefore the above hypothesis is false and f is holomorphic on $\mathbb{C}$. The converse is obvious.

Remark 6. To compare the above theorem and some results of [2], observe that we can not write $(w+2f)(w-f)$ on the form $p(w-f)$, where p is a holomorphic polynomial on $\mathbb{C}$ and $f:\mathbb{C}\to\mathbb{C}$, $f\neq 0$. But if q is the following holomorphic polynomial on $\mathbb{C}^2$, defined by $q(\xi,w)=(w+2\xi)(w-\xi)$, for $(\xi,w)\in\mathbb{C}^2$, we can write $(w+2f)(w-f)=q(f,w)$. Denote $\psi(\xi,w)=|q(\overline{\xi},w)|$. Then ψ is not psh on $\mathbb{C}^2$. (We say in this case that $|q|$ characterize holomorphic functions).

Proposition 6. *Let $f_1,g_1,f_2,g_2,f_3,g_3:\mathbb{C}^n\to\mathbb{C}$ be holomorphic functions. Define $u(z,w)=|w-f_1(z)-\overline{g_1}(z)||w-f_2(z)-\overline{g_2}(z)||w-f_3(z)-\overline{g_3}(z)|$, for $(z,w)\in\mathbb{C}^n\times\mathbb{C}$.
The following conditions are equivalent
(I) u is psh on $\mathbb{C}^n\times\mathbb{C}$;
(II) We have only case 1, or case 2.
Case 1. $(g_1+g_2+g_3)$ is constant,
$(f_1+\overline{g_1})(f_2+\overline{g_2})+(f_1+\overline{g_1})(f_3+\overline{g_3})+(f_2+\overline{g_2})(f_3+\overline{g_3})$ is holomorphic on $\mathbb{C}^n$ and $(f_1+\overline{g_1})(f_2+\overline{g_2})(f_3+\overline{g_3})$ is prh on $\mathbb{C}^n$.
Case 2. $(g_1+g_2+g_3)$ is non constant and
$(f_1+\overline{g_1})(f_2+\overline{g_2})+(f_1+\overline{g_1})(f_3+\overline{g_3})+(f_2+\overline{g_2})(f_3+\overline{g_3})=\frac{1}{3}(f_1+f_2+f_3+\overline{g_1}+\overline{g_2}+\overline{g_3})^2$ on $\mathbb{C}^n$.*

Proof. Obvious by the preceding theorem. In general, we have the following problems.

Problem 1. Let $n,N\in\mathbb{N}$, $N\geq 2$, D is a domain of $\mathbb{C}^n$. Find all the analytic functions $g_1,...,g_N:D\to\mathbb{C}$, such that u is psh on $D\times\mathbb{C}$. Here $u(z,w)=|w-\overline{g_1}(z)|...|w-\overline{g_N}(z)|$, for $(z,w)\in D\times\mathbb{C}$.

Problem 2. Let $v(z,w)=|f_1(z)-\overline{g_1}(w)|...|f_N(z)-\overline{g_N}(w)|$, $f_1,...,f_N:\mathbb{C}^n\to\mathbb{C}$ and $g_1,...,g_N:\mathbb{C}^m\to\mathbb{C}$ be $2N$ holomorphic functions, $N\geq 2$ and $(z,w)\in\mathbb{C}^n\times\mathbb{C}^m$. Find all the conditions described by $f_1,...,f_N,g_1,...,g_N$ such that v is convex on $\mathbb{C}^n\times\mathbb{C}^m$.

Problem 3. Put $v=|g_1-\varphi_1|...|g_N-\varphi_N|$, where $g_1,...,g_N:\mathbb{C}^m\to\mathbb{C}$ and $\varphi_1,...,\varphi_N:\mathbb{C}^n\to\mathbb{C}$ be $2N$ prh functions. Establish all the conditions satisfying by $g_1,...,g_N$, $\varphi_1,...,\varphi_N$ such that v is psh on $\mathbb{C}^n\times\mathbb{C}^m$.

Problem 4. Let $a_1,...,a_N\in\mathbb{C}^m$, $\varphi_1,...,\varphi_N:\mathbb{C}^n\to\mathbb{C}$ be holomorphic functions, $N\geq 2$. Put $v(z,w)=|<w/a_1>-\overline{\varphi_1}(z)|...|<w/a_N>-\overline{\varphi_N}(z)|$, $(z,w)\in\mathbb{C}^n\times\mathbb{C}^m$. Characterize $a_1,...,a_N$, $\varphi_1,...,\varphi_N$, such that v is psh on $\mathbb{C}^n\times\mathbb{C}^m$.

Remark 7. Let $v_N(z,w)=|w-\overline{\varphi_1}(z)|...|w-\overline{\varphi_N}(z)|$, $\varphi_1,...,\varphi_N:\mathbb{C}^n\to\mathbb{C}$ be holomorphic functions, $N\geq 2$, $(z,w)\in\mathbb{C}^n\times\mathbb{C}$.

Consider the problem (E_N) : v_N is psh on $\mathbb{C}^n \times \mathbb{C}$.
A technical key for the study of the problem (E_N) is a consequence of the classical cases (E_2), (E_3) and (E_4) which are proved. Note that if $|w-\overline{\varphi_1}||w-\overline{\varphi_2}|$ is psh then for each holomorphic function φ_3, the new function $|w-\overline{\varphi_1}||w-\overline{\varphi_2}||w-\overline{\varphi_3}|$ is not psh on $\mathbb{C}^n \times \mathbb{C}$ if φ_1 and φ_2 are nonconstant functions and $\varphi_3 \neq \varphi_1$, or $\varphi_3 \neq \varphi_2$.
The converse. Let $u(z,w) = |w-\overline{\varphi_1}(z)||w-\overline{\varphi_2}(z)||w-\overline{\varphi_3}(z)|$. Suppose that u is psh on $\mathbb{C}^n \times \mathbb{C}$ and φ_j is non constant, $1 \leq j \leq 3$. Then $|w-\overline{\varphi_1}||w-\overline{\varphi_2}|$, $|w-\overline{\varphi_1}||w-\overline{\varphi_3}|$ and $|w-\overline{\varphi_2}||w-\overline{\varphi_3}|$ are not psh if $(\varphi_1+\varphi_2+\varphi_3)$ is constant, $\varphi_1 \neq \varphi_2$, $\varphi_1 \neq \varphi_3$ and $\varphi_2 \neq \varphi_3$.
Recall that u is psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $(\varphi_1+\varphi_2+\varphi_3)$ is constant and $(\varphi_1\varphi_2+\varphi_1\varphi_3+\varphi_2\varphi_3)$ is constant, or $(\varphi_1+\varphi_2+\varphi_3)$ is nonconstant and $\varphi_1=\varphi_2=\varphi_3$ on $\mathbb{C}^n$.

Remark 8. Consider the functions $g_1(z)=z$, $g_2(z)=-z$, $g_3(z)=iz$, $g_4(z)=-iz$, for $z \in \mathbb{C}$. g_1, g_2, g_3 and g_4 are holomorphic functions on $\mathbb{C}$. Let $(z,w) \in \mathbb{C}^2$.
$v(z,w) = |w-\overline{g_1}(z)||w-\overline{g_2}(z)||w-\overline{g_3}(z)||w-\overline{g_4}(z)| = |w^2-(\overline{z})^2||w^2+(\overline{z})^2| = |w^4-(\overline{z})^4|$.
$v=|h|$, where $h:\mathbb{C}^2 \to \mathbb{C}$ is prh. Then v is psh on $\mathbb{C}^2$. But v_1, v_2, v_3 and v_4 are not psh functions on $\mathbb{C}^2$, where

$$v_1(z,w) = |w-\overline{g_1}(z)||w-\overline{g_2}(z)||w-\overline{g_3}(z)|,$$

$$v_2(z,w) = |w-\overline{g_1}(z)||w-\overline{g_2}(z)||w-\overline{g_4}(z)|,$$

$$v_3(z,w) = |w-\overline{g_1}(z)||w-\overline{g_3}(z)||w-\overline{g_4}(z)|,$$

$$v_4(z,w) = |w-\overline{g_2}(z)||w-\overline{g_3}(z)||w-\overline{g_4}(z)|.$$

Note that a precise study of the plurisubharmonicity of the two functions ψ_1 and ψ_2 extends some interesting and sharp results in the framework of a slightly different direction. We can study the complex nature of the function $\psi_3(z,w) = |w-\overline{g_1}(z)|...|w-\overline{g_N}(z)|$, where ($N=2^k$, or $N=3\times 2^k$, $k\in\mathbb{N}$, $k\geq 2$), $g_1,...,g_N:\mathbb{C}^n \to \mathbb{C}$ are holomorphic functions, $\psi_1(z,w) = \prod_{1\leq j\leq 4}|w-\overline{\varphi_j}(z)|$, $\psi_2(z,w) = \prod_{1\leq j\leq 8}|w-\overline{\varphi_j}(z)|$ and $\varphi_j:\mathbb{C}^n \to \mathbb{C}$ is a holomorphic function, $1\leq j\leq 8$.
In the sequel, the next result gives the exact characterization according to algebraic methods in the theory of holomorphic polynomials and related topics. We have

Theorem 15. *Let $\varphi_1,\varphi_2,\varphi_3,\varphi_4 : D \to \mathbb{C}$ be four holomorphic functions, D is a domain of $\mathbb{C}$. Put $u(z,w) = |w-\overline{\varphi_1}(z)||w-\overline{\varphi_2}(z)||w-\overline{\varphi_3}(z)||w-\overline{\varphi_4}(z)|$, $(z,w)\in D\times\mathbb{C}$. Let $v=u^2$. The following conditions are equivalent*
(I) u is psh on $D\times\mathbb{C}$;
(II) We have the following cases.
Case 1. $\frac{\partial^2 v}{\partial\overline{z}\partial w}=0$ *on* $D\times\mathbb{C}$.
Case 2. $\frac{\partial^2 v}{\partial\overline{z}\partial w}\neq 0$ *on* $D\times\mathbb{C}$ *and we have the following two conditions.*
Step 1. $(\sum_{j=1}^{4}\varphi_j)$ *is nonconstant and* $\varphi_1=\varphi_2=\varphi_3=\varphi_4$ *on* D.

Step 2. $(\sum_{j=1}^{4}\varphi_j)$ *is constant on* D *and we have the following assertion.*
There exists j_1, j_2, j_3, j_4, *satisfying* $j_1 < j_2$, $j_3 < j_4$, $\{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$, $\varphi_{j_1} = \varphi_{j_2}$, $\varphi_{j_3} = \varphi_{j_4}$ *and the function* $\varphi_{j_1}\varphi_{j_3}$ *is nonconstant on* D.

Proof. (I) implies (II). Let $(z, w) \in D \times \mathbb{C}$. We have
$v(z, w) = |w^4 - \overline{s_1}(z)w^3 + \overline{s_2}(z)w^2 - \overline{s_3}(z)w + \overline{s_4}(z)|^2$.
$s_1 = \sum_{j=1}^{4}\varphi_j$, $s_2 = \sum_{1\le j<k\le 4}\varphi_j\varphi_k$, $s_3 = \sum_{1\le j<k<s\le 4}\varphi_j\varphi_k\varphi_s$, $s_4 = \varphi_1\varphi_2\varphi_3\varphi_4$.
s_1, s_2, s_3 and s_4 are holomorphic functions on D.
v is a function of class C^∞ and psh on $D \times \mathbb{C}$.
$\frac{\partial^2 v}{\partial\overline{w}\partial w}(z, w) = |4w^3 - 3\overline{s_1}(z)w^2 + 2\overline{s_2}(z)w - \overline{s_3}(z)|^2 \ge 0$.
$\frac{\partial^2 v}{\partial\overline{z}\partial z}(z, w) = | - \overline{s_1'}(z)w^3 + \overline{s_2'}(z)w^2 - \overline{s_3'}(z)w + \overline{s_4'}(z)|^2 \ge 0$ and $\frac{\partial^2 v}{\partial\overline{z}\partial w}(z, w) = (-3\overline{s_1'}(z)w^2 + 2\overline{s_2'}(z)w - \overline{s_3'}(z))\overline{[w^4 - \overline{s_1}(z)w^3 + \overline{s_2}(z)w^2 - \overline{s_3}(z)w + \overline{s_4}(z)]}$.
Since v is psh on $D \times \mathbb{C}$, then we have the inequality

$$(E):\ |\frac{\partial^2 v}{\partial\overline{z}\partial w}(z, w)|^2 \le \frac{\partial^2 v}{\partial\overline{z}\partial z}(z, w)\frac{\partial^2 v}{\partial\overline{w}\partial w}(z, w)$$

for each $(z, w) \in D \times \mathbb{C}$.
Put

$$q_1(w) = (w - \overline{\varphi_1}(z))(w - \overline{\varphi_2}(z))(w - \overline{\varphi_3}(z))(w - \overline{\varphi_4}(z)),$$
$$q_2(w) = -3\overline{s_1'}(z)w^2 + 2\overline{s_2'}(z)w - \overline{s_3'}(z),$$
$$q_3(w) = 4w^3 - 3\overline{s_1}(z)w^2 + 2\overline{s_2}(z)w - \overline{s_3}(z),$$
$$q_4(w) = -\overline{s_1'}(z)w^3 + \overline{s_2'}(z)w^2 - \overline{s_3'}(z)w + \overline{s_4'}(z).$$

q_1, q_2, q_3 and q_4 are holomorphic polynomials on $\mathbb{C}$, for each fixed z on D.
We have $q_1' = q_3$ and $q_4' = q_2$. By the inequality (E) we have then $|q_1q_2| \le |q_3q_4|$ on $\mathbb{C}$.
Case 1. $q_2(w) = 0$, for every $w \in \mathbb{C}$ and for any $z \in D$.
Then s_1, s_2 and s_3 are constant functions on D. Therefore $\frac{\partial^2 v}{\partial\overline{z}\partial w} = 0$ on $D \times \mathbb{C}$. Thus we have

$$u(z, w) = |w^4 + c_1w^3 + c_2w^2 + c_3w + \overline{\varphi_1}(z)\overline{\varphi_2}(z)\overline{\varphi_3}(z)\overline{\varphi_4}(z)|$$

where $c_1, c_2, c_3 \in \mathbb{C}$. Therefore $u = |h|$, when h is a pluriharmonic (prh) function on $D \times \mathbb{C}$. Consequently, u is psh on $D \times \mathbb{C}$.
Case 2. $q_2 \ne 0$ on $\mathbb{C}$.
Now fix $z \in D$, such that $[-3\overline{s_1'}(z)w^2 + 2\overline{s_2'}(z)w - \overline{s_3'}(z)] \ne 0$. Since $q_1q_2 \ne 0$ and the inequality (E), there exists $c \in \mathbb{C}\backslash\{0\}$ such that $q_3q_4 = cq_1q_2$ on $\mathbb{C}$.
Step 1. $s_1' \ne 0$ on $\mathbb{C}$.
Then $c = \frac{4}{3}$. We have $A = \{\overline{\varphi_1}(z), \overline{\varphi_2}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$ is the set of all zeros of the analytic polynomial q_1. Assume that the cardinality of A is equal to 4. Observe that because of the property of the order of multiplicity of zeros of a polynomial and the

relation $q_3q_4 = \frac{4}{3}q_1q_2$, we have $\overline{\varphi_1}(z)$, $\overline{\varphi_2}(z)$, $\overline{\varphi_3}(z)$, $\overline{\varphi_4}(z)$ are distinct zeros of q_4. Therefore $\deg(q_4) \geq 4$. A contradiction. Consequently, the cardinal of the subset A is less than or equal 3.

Without loss of generality, we assume that $\varphi_1 = \varphi_2$. Assume that $\varphi_1 \neq \varphi_3$. We have $\overline{\varphi_1}(z)$ is a zero of q_3 and q_4. Note that $\{\overline{\varphi_1}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$ is exactly the set of zeros of the holomorphic polynomial q_4. Let $\overline{w_1}$ and $\overline{w_2}$ the two zeros of the polynomial q_2. Indeed, for instance, using possible relations between all the coefficients of a holomorphic polynomial and its zeros, we have then

$$\overline{w_1} + \overline{w_2} = \frac{2}{3}\frac{\overline{s_2'}(z)}{\overline{s_1'}(z)} = \frac{2}{3}(\overline{\varphi_1}(z) + \overline{\varphi_3}(z) + \overline{\varphi_4}(z)).$$

$$\overline{w_1 w_2} = \frac{\overline{s_3'}(z)}{3\overline{s_1'}(z)} = \frac{1}{3}(\overline{\varphi_1}(z)\overline{\varphi_3}(z) + \overline{\varphi_1}(z)\overline{\varphi_4}(z) + \overline{\varphi_3}(z)\overline{\varphi_4}(z)).$$

Assume that $\overline{\varphi_1}(z)$, $\overline{\varphi_3}(z)$ and $\overline{\varphi_4}(z)$ are zeros of q_4 of order 1. Then $\overline{w_1}$ and $\overline{w_2}$ are not zeros of q_4.

$\overline{w_1}$ and $\overline{w_2}$ are zeros of q_3.

Therefore $\overline{\varphi_1}(z) + \overline{w_1} + \overline{w_2} = \frac{3}{4}\overline{s_1}(z) = \frac{3}{4}(2\overline{\varphi_1}(z) + \overline{\varphi_3}(z) + \overline{\varphi_4}(z))$. Then $w_1 + w_2 = \frac{2}{3}(\varphi_1(z) + \varphi_3(z) + \varphi_4(z))$.

$w_1 + w_2 = \frac{1}{2}\varphi_1(z) + \frac{3}{4}(\varphi_3(z) + \varphi_4(z))$. Thus $\varphi_3(z) + \varphi_4(z) = 2\varphi_1(z)$ and then

$$\begin{aligned} w_1 + w_2 &= 2\varphi_1(z). \\ \overline{w_1 w_2} &= \frac{\overline{s_3'}(z)}{3\overline{s_1'}(z)} = \frac{1}{3}(\overline{\varphi_1}(z)\overline{\varphi_3}(z) + \overline{\varphi_1}(z)\overline{\varphi_4}(z) + \overline{\varphi_3}(z)\overline{\varphi_4}(z)) \\ &= \frac{1}{3}[2(\overline{\varphi_1}(z))^2 + \overline{\varphi_3}(z)\overline{\varphi_4}(z)]. \end{aligned}$$

We have also

$$w_1\varphi_1(z) + w_2\varphi_1(z) + w_1w_2 = \frac{2s_2(z)}{4} = \frac{s_2(z)}{2} =$$

$$\frac{1}{2}(\varphi_1^2(z) + 2\varphi_1(z)\varphi_3(z) + 2\varphi_1(z)\varphi_4(z) + \varphi_3(z)\varphi_4(z)) = \frac{1}{2}(5\varphi_1^2(z) + \varphi_3(z)\varphi_4(z))$$

$$= (w_1 + w_2)\varphi_1 + w_1w_2.$$

Therefore,

$$2\varphi_1^2(z) + w_1w_2 = \frac{1}{2}(5\varphi_1^2(z) + \varphi_3(z)\varphi_4(z)).$$

$$w_1w_2 = \frac{1}{2}\varphi_1^2(z) + \frac{1}{2}\varphi_3(z)\varphi_4(z) = \frac{1}{3}(2\varphi_1^2(z) + \varphi_3(z)\varphi_4(z)).$$

Thus $3\varphi_1^2(z) + 3\varphi_3(z)\varphi_4(z) = 4\varphi_1^2(z) + 2\varphi_3(z)\varphi_4(z)$. Then $\varphi_3(z)\varphi_4(z) = \varphi_1^2(z)$. Since $\varphi_3 + \varphi_4(z) = 2\varphi_1(z)$. Thus $\varphi_3(z) = \varphi_4(z) = \varphi_1(z)$. A contradiction.

Assume now that $\overline{\varphi_1}(z)$, $\overline{\varphi_3}(z)$ and $\overline{\varphi_4}(z)$ are not zeros of q_4 of order 1. Recall that $\{\overline{\varphi_1}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$ is exactly the set of zeros of q_4. Since $\varphi_1(z) \neq \varphi_3(z)$, then $\varphi_1(z) = \varphi_4(z)$.

Because if $\varphi_1(z) \neq \varphi_4(z)$, then $\varphi_3(z) = \varphi_4(z)$. Since $\varphi_1(z) = \varphi_2(z) \neq \varphi_3(z)$, then $u(\xi, w) = |w - \overline{\varphi_1}(\xi)|^2 |w - \overline{\varphi_3}(\xi)|^2$, for each $(\xi, w) \in G = D(z, R) \times \mathbb{C}$, where $R > 0$ satisfying $D(z, R) \subset D$. Since u is a function of class C^∞ and psh on the domain G, we can prove that we have the condition $\varphi_1 = \varphi_3$ on $D(z, R)$, or $(\varphi_1 + \varphi_3) = \frac{s_1}{2}$ is constant on $D(z, R)$.
Now since s_1 is holomorphic nonconstant on D, then s_1 is nonconstant on the open Euclidean disc $D(z, R)$. It follows that $\varphi_1 = \varphi_3$ on $D(z, R)$. A contradiction, because $\varphi_1(z) \neq \varphi_3(z)$. Consequently, $\varphi_1(z) = \varphi_4(z)$. Therefore $\overline{\varphi_1}(z)$ is a zero of q_2 of order 1. Assume that $\varphi_1(z) = w_1$. We have $\varphi_1(z) = \varphi_2(z) = \varphi_4(z)$, $\varphi_1(z) \neq \varphi_3(z)$. It follows that $\overline{\varphi_1}(z)$ is a zero of q_3 of order 2.
$\overline{\varphi_3}(z)$ is not a zero of q_3.
Now we use the classical relations between all the coefficients of a polynomial and its zeros, we have $\varphi_1(z) + w_2 = \frac{2s_2'(z)}{3s_1'(z)} = \frac{2}{3}(2\varphi_1(z) + \varphi_3(z))$. Also $2\varphi_1(z) + w_2 = \frac{3}{4}(3\varphi_1(z) + \varphi_3(z))$ and $\varphi_1(z) + w_2 = \frac{2}{3}(2\varphi_1(z) + \varphi_3(z))$. Then $\varphi_1(z) + \frac{2}{3}(2\varphi_1(z) + \varphi_3(z)) = \frac{3}{4}(3\varphi_1(z) + \varphi_3(z))$. Thus, $12\varphi_1(z) + 8(2\varphi_1(z) + \varphi_3(z)) = 9(3\varphi_1(z) + \varphi_3(z))$. Consequently, $\varphi_1(z) = \varphi_3(z)$. A contradiction. It follows that the assumption $\varphi_1(z) \neq \varphi_3(z)$ is impossible. Consequently, $\varphi_1(z) = \varphi_2(z) = \varphi_3(z)$.
Now assume that $\varphi_4(z) \neq \varphi_1(z)$. Let $\overline{w_0}$ the zero of q_2, $w_0 \neq \varphi_1(z)$. Note that $\overline{\varphi_1}(z)$ is a zero of the polynomial q_2 because $\overline{\varphi_1}(z)$ is a zero of q_4 of order 2.
$\overline{\varphi_1}(z)$ is a zero of q_1 of order 3.
Therefore $\overline{\varphi_1}(z)$ is a zero of q_3 of order 2. Consequently, $\overline{w_0}$ is a zero of q_3 of order 1. We have $w_0 + \varphi_1(z) = \frac{2s_2'(z)}{3s_1'(z)} = \frac{2}{3}(2\varphi_1(z) + \varphi_4(z))$. Also $2\varphi_1(z) + w_0 = \frac{3}{4}s_1(z) = \frac{3}{4}(3\varphi_1(z) + \varphi_4(z))$. Therefore, we have $\varphi_1(z) + \frac{2}{3}(2\varphi_1(z) + \varphi_4(z)) = \frac{3}{4}(3\varphi_1(z) + \varphi_4(z))$. Thus $\varphi_1(z) = \varphi_4(z)$. A contradiction. Consequently, the assumption $\varphi_1(z) \neq \varphi_4(z)$ is impossible. We conclude that $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ on D.
Step 2. s_1 is constant on D.
Let $(z, w) \in D \times \mathbb{C}$, such that $\frac{\partial^2 v}{\partial \bar{z} \partial w}(z, w) \neq 0$. Assume that $s_2'(z) \neq 0$. We have $q_1 q_2 = c q_3 q_4$, where $c = \frac{1}{2}$. Let $\overline{w_0} = \frac{\overline{s_3'(z)}}{2\overline{s_2'(z)}}$ be the only zero of q_2. Note that $\{\overline{\varphi_1}(z), \overline{\varphi_2}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$ is the set of zeros of the holomorphic polynomial q_1 on $\mathbb{C}$. If for example $\overline{\varphi_1}(z)$ is a zero of q_1 of order 1. Then $\overline{\varphi_1}(z)$ is not a zero of $q_3 = q_1'$. Since now $q_1 q_2 = \frac{1}{2} q_3 q_4$, then $\overline{\varphi_1}(z)$ is a zero of q_4.
Now if $\overline{\varphi_1}(z)$ is a zero of q_1 of order 2. Then $\overline{\varphi_1}(z)$ is a zero of $q_3 = q_1'$ of order 1. By the fundamental relation $q_1 q_2 = \frac{1}{2} q_3 q_4$, we obtain $\overline{\varphi_1}(z)$ is a zero of q_4. We conclude that the set of zeros of q_4 is $\{\overline{\varphi_1}(z), \overline{\varphi_2}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$.
Since now $\deg(q_4) = 2$ (because $s_2'(z) \neq 0$), then there exists j_1, j_2, j_3, j_4, $\{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$, such that $\varphi_{j_1} = \varphi_{j_2} = \varphi_{j_3} \neq \varphi_{j_4}$ on D, or $\varphi_{j_1} = \varphi_{j_2}$ and $\varphi_{j_3} = \varphi_{j_4}$ on D.
Suppose that we have $\varphi_1 = \varphi_2 = \varphi_3 \neq \varphi_4$. Then $\overline{\varphi_1}(z)$ is a zero of q_1 of order 3. $\overline{\varphi_1}(z)$ is a zero of q_3 of order 2.
$\overline{w_0}$ is a zero of q_3 of order 1.
$\overline{\varphi_4}(z)$ is not a zero of q_3.
We have

$$2\varphi_1(z) + w_0 = \frac{3}{4}s_1(z) = \frac{3}{4}(3\varphi_1(z) + \varphi_4(z)).$$

$$w_0 = \frac{s_3'(z)}{2s_2'(z)} = \frac{1}{2}(\varphi_1(z) + \varphi_4(z)).$$

Thus $2\varphi_1(z) + \frac{1}{2}(\varphi_1(z) + \varphi_4(z)) = \frac{3}{4}(3\varphi_1(z) + \ \varphi_4(z))$. Therefore $\frac{5}{2}\varphi_1(z) + \frac{1}{2}\varphi_4(z) = \frac{9}{4}\varphi_1(z) + \frac{3}{4}\varphi_4(z)$. Then $\varphi_1(z) = \varphi_4(z)$. A contradiction. Consequently, the above assumption is impossible. It follows that $\varphi_{j_1} = \varphi_{j_2}$ and $\varphi_{j_3} = \varphi_{j_4}$ on D, (for example). We suppose without loss of generality that $j_1 < j_2$ and $j_3 < j_4$. Then $u(z,w) = |w - \overline{\varphi_{j_1}}(z)|^2|w - \overline{\varphi_{j_3}}(z)|^2$, for $(z,w) \in D \times \mathbb{C}$.
Actually, we observe that $\varphi_{j_1} = \varphi_{j_3}$ on D, or $(\varphi_{j_1} + \varphi_{j_3})$ is constant on D. Suppose that $\varphi_{j_1} = \varphi_{j_3}$ on D. Then $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ on D. Since $s_1' = 0$ on D, then φ_1 is constant on D. Thus $\frac{\partial^2 v}{\partial \bar{z} \partial w} = 0$ on $D \times \mathbb{C}$. A contradiction. Consequently, $(\varphi_{j_1} + \varphi_{j_3})$ is constant on D and observe that the product $\varphi_{j_1}\varphi_{j_3}$ is nonconstant on D.
Assume that $s_2' = 0$ on D. Then $s_3' \neq 0$ on D, because $\frac{\partial^2 v}{\partial \bar{z} \partial w} \neq 0$ on $D \times \mathbb{C}$. The set of zeros of q_4 is $\{\overline{\varphi_1}(z), \overline{\varphi_2}(z), \overline{\varphi_3}(z), \overline{\varphi_4}(z)\}$. Since $\deg(q_4) = 1$, then $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ on D.
$s_1' = 0$ on D implies that φ_1 is constant on D. Therefore, $\frac{\partial^2 v}{\partial \bar{z} \partial w} = 0$ on $D \times \mathbb{C}$. It follows that this case is impossible.
(II) implies (I). Obvious.

Remark 9 . Let $F : \mathbb{C}^2 \to \mathbb{C}$ be holomorphic, $a_1, a_2, a_3, a_4 \in \mathbb{C}^4$ and $b_1, b_2, b_3, b_4 \in \mathbb{C}$. Define

$$\begin{aligned} v(z,w) &= |[w - \overline{< F(z)/a_1 >} - b_1][w - \overline{< F(z)/a_2 >} - b_2] \cdot \\ &\quad \cdot [w - \overline{< F(z)/a_3 >} - b_3][w - \overline{< F(z)/a_4 >} - b_4]|, \end{aligned}$$

for $(z,w) \in \mathbb{C}^2 \times \mathbb{C}$. We can characterize all the conditions on a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, which ensure technical hypothesis for the plurisubharmonicity of v. Indeed, we have the following of various behaviour.

Theorem 16. *Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : D \to \mathbb{C}$ be holomorphic functions, D is a domain of $\mathbb{C}^n$, $n \geq 1$.*
Put $u(z,w) = |w - \overline{\varphi_1}(z)||w - \overline{\varphi_2}(z)||w - \overline{\varphi_3}(z)||w - \overline{\varphi_4}(z)|$, $(z,w) \in D \times \mathbb{C}$.
Let $v = u^2$, $s_1 = \sum_{j=1}^{4} \varphi_j$, $s_2 = \sum_{1 \leq j < k \leq 4} \varphi_j \varphi_k$, $s_3 = \sum_{1 \leq j < k < s \leq 4} \varphi_j \varphi_k \varphi_s$, $s_4 = \varphi_1\varphi_2\varphi_3\varphi_4$, ($s_1, s_2, s_3, s_4$ are holomorphic functions on D).
The following assertions are equivalent
(I) u (respectively v) is psh on $D \times \mathbb{C}$;
(II) We have the following three cases.
Case 1. s_1, s_2 and s_3 are constant on D.
Case 2. s_1 is nonconstant on D and $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$ on D.
Case 3. s_1 is constant on D, s_2 is nonconstant on D and there exits j_1, j_2, j_3, j_4, $\{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$, $j_1 < j_2$, $j_3 < j_4$, with $\varphi_{j_1} = \varphi_{j_2}$ and $\varphi_{j_3} = \varphi_{j_4}$ on D.

Proof. Obvious by the above theorem.

Example. Let $a_1, a_2, a_3, a_4 \in \mathbb{C}^2$, $A_1, A_2, A_3, A_4 \in \mathbb{C}^m$ and $F : \mathbb{C}^n \to \mathbb{C}^2$ be a holomorphic function, $n, m \geq 1$. Define

$$\begin{aligned}\psi(z,w) &= |<w/A_1> - \overline{<F(z)/a_1>}||<w/A_2> - \overline{<F(z)/a_2>}| \cdot \\ &\cdot |<w/A_3> - \overline{<F(z)/a_3>}||<w/A_4> - \overline{<F(z)/a_4>}|,\end{aligned}$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}^m$.
In a slightly different direction, we can show all the conditions formulated by the constants a_1, a_2, a_3, a_4, A_1, A_2, A_3, A_4, which characterize the plurisubharmonicity of ψ.

Example. Let $N \geq 2$ and $p(\xi_1,...,\xi_N,w) = (w-\xi_1)...(w-\xi_N)$, for $(\xi_1,...,\xi_N) \in \mathbb{C}^N$, $w \in \mathbb{C}$. Define $F(\xi_1,...,\xi_N,w) = |p(\overline{\xi_1},...,\overline{\xi_N},w)|$. Then for each Euclidean open ball $B(a,R) \subset \mathbb{C}^N$, $(a \in \mathbb{C}^N, R > 0)$, the function F is not psh on $B(a,R) \times \mathbb{C}$.

Remark 10. (I) Let $g_1(z) = z^2$, $g_2(z) = -z^2$, $g_3(z) = iz^2$, $g_4(z) = -iz^2$, $z \in \mathbb{C}$. g_1, g_2, g_3 and g_4 are holomorphic functions on $\mathbb{C}$. Let

$$u_1(z,w) = |w - \overline{g_1}(z)||w - \overline{g_2}(z)||w - \overline{g_3}(z)|,$$

$$u_2(z,w) = |w - \overline{g_1}(z)||w - \overline{g_2}(z)||w - \overline{g_4}(z)|,$$

$$u_3(z,w) = |w - \overline{g_1}(z)||w - \overline{g_3}(z)||w - \overline{g_4}(z)|,$$

$$u_4(z,w) = |w - \overline{g_2}(z)||w - \overline{g_3}(z)||w - \overline{g_4}(z)|,$$

$$u(z,w) = |w - \overline{g_1}(z)||w - \overline{g_2}(z)||w - \overline{g_3}(z)||w - \overline{g_4}(z)|, \ (z,w) \in \mathbb{C}^2.$$

We have u_1, u_2, u_3 and u_4 are not psh functions on $\mathbb{C}^2$. But u is psh on $\mathbb{C}^2$.
(II) $g_1(z) = g_2(z) = z + 1$, $g_3(z) = g_4(z) = -z + 1$, $z \in \mathbb{C}$.
g_1, g_2, g_3 and g_4 are holomorphic functions on $\mathbb{C}$.
$(g_1 + g_2 + g_3 + g_4)$ is constant on $\mathbb{C}$.
$(g_1g_2 + g_1g_3 + g_1g_4 + g_2g_3 + g_2g_4 + g_3g_4)$ is non constant on $\mathbb{C}$.
Let $(z,w) \in \mathbb{C}^2$. Put $u(z,w) = |w-\overline{g_1}(z)||w-\overline{g_2}(z)||w-\overline{g_3}(z)||w-\overline{g_4}(z)| = |w^4 - 4w^3 + [6-2(\overline{z})^2]w^2 - 4w[1-(\overline{z})^2] + [1-(\overline{z})^2]^2|$. Observe that $(g_1g_2g_3+g_1g_2g_4+g_1g_3g_4+g_2g_3g_4)$ is nonconstant on $\mathbb{C}$. But u is psh on $\mathbb{C}^2$, because

$$u(z,w) = |w - 1 - \overline{z}|^2|w - 1 + \overline{z}|^2 = |(w-1)^2 - (\overline{z})^2|^2 = |h|^2,$$

where h is a prh function on $\mathbb{C}^2$.

Question 8. Let $N_1,...,N_k,s_1,m_1,...,s_t,m_t \in \mathbb{N}\backslash\{0\}$, $k,t \geq 1$ and $g_1,...,g_k$, $\theta_1,...,\theta_t : \mathbb{C}^n \to \mathbb{C}$ be prh functions. Put

$$u(z,w) = |w - g_1(z)|^{N_1}...|w - g_k(z)|^{N_k}|w^{s_1} - \theta_1^{m_1}(z)|...|w^{s_t} - \theta_t^{m_t}(z)|,$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Find conditions $g_1,...,g_k,\theta_1,...,\theta_t$ should satisfy so that u is psh on $\mathbb{C}^n \times \mathbb{C}$.

Question 9. Let $N \in \mathbb{N}\backslash\{0,1\}$ and $A_0,...,A_{N-1} \in \mathbb{C}$. Define $v(z,w) = |w^N + A_{N-1}w^{N-1}\overline{z} + ... + A_1w\overline{z^{N-1}} + A_0\overline{z^N}|$. Find all conditions on N, $A_0,...,A_{N-1}$ such that v is psh on $\mathbb{C}^2$.

Conclude that we can characterize all the holomorphic polynomials q on $\mathbb{C}^2$, such that F is psh on $\mathbb{C}^2$, where $F(z,w) = |q(\overline{z}, w)|$ for $(z,w) \in \mathbb{C}^2$.
Let p be a holomorphic polynomial on $\mathbb{C}^2$. Put $F_1(z,w) = |p(\overline{z}, w)|$ and $F_2(z,w) = |p(z, \overline{w})|$, for $(z,w) \in \mathbb{C}^2$. Moreover, thanks to the above characterization, we can prove that F_1 is psh on $\mathbb{C}^2$ if and only if F_2 is psh on $\mathbb{C}^2$.
In the following question, we recall some properties and sharp results in the framework of complex analysis of the appeared function θ, defined by $\theta(z,w) = (w + \overline{z})^N$, for $N \in \mathbb{N}\backslash\{0,1\}$ and $(z,w) \in \mathbb{C}^2$.

Question 10. Let $N \in \mathbb{N}\backslash\{0,1\}$, $A \in \mathbb{C}\backslash\{0\}$, $(B_1, ..., B_n) \in \mathbb{C}^n\backslash\{0\}$ and $s \in [1, +\infty[$. Let $g, f_0, ..., f_{N-2} : D \to \mathbb{C}$ be continuous functions, where D is a domain on $\mathbb{C}^n$. Define $\psi(z,w) = |(Aw + B_1\overline{z_1} + ... + B_n\overline{z_n})^N + g(z)|^s$ and $\varphi(z,w) = |(Aw + B_1\overline{z_1} + ... + B_n\overline{z_n})^N + f_{N-2}(z)w^{N-2} + ... + f_0(z)|$, for $(z,w) = (z_1, ..., z_n, w) \in D \times \mathbb{C}$. Assume that ψ is psh on $D \times \mathbb{C}$. Prove that $g = 0$ on D. Suppose that φ is psh on $D \times \mathbb{C}$. Prove that $f_{N-2} = ... = f_0 = 0$.

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