

Karakostas Fixed Point Theorem and Semilinear Neutral Differential Equations with Impulses and Nonlocal Conditions

Hugo Leiva, Lenin Riera and Sebastián Lalvay

ABSTRACT: This paper is concerned with the existence and uniqueness of solutions for a semilinear neutral differential equation with impulses and nonlocal conditions. First, we assume that the nonlinear terms are locally Lipschitz, and to achieve the existence of solutions, Karakostas Fixed Point Theorem is applied. After that, under some additional conditions, the uniqueness is proved as well. Next, assuming some bound on the nonlinear terms the global existence is proved by applying a generalization of Gronwall inequality for impulsive differential equations. Then, we suppose stronger hypotheses on the nonlinear functions, such as globally Lipschitz conditions, that allow us to apply Banach Fixed Point Theorem to prove the existence and uniqueness of solutions. Finally, we present an example as an application of our method.

AMS Subject Classification: 93B05, 93C10.

Keywords and Phrases: Semilinear neutral differential equations; Impulses; Delay; Nonlocal conditions; Karakostas Fixed Point Theorem.

1. Introduction and Preliminaries

This work is devoted to study the existence of solutions for the following semilinear neutral differential equation with impulses and nonlocal conditions.

$$\begin{cases} \frac{d}{dt}[z(t) - f_{-1}(t, z_t)] = A_0(t)z(t) + f_1(t, z_t), & t \neq t_k, \quad t \in [0, \tau], \\ z(\theta) + h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(\theta) = \eta(\theta), & \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), & k = 1, 2, \dots, p, \end{cases} \quad (1.1)$$

where $A_0(t)$ is a $n \times n$ continuous matrix, the functions f_{-1}, f_1 , and h are smooth enough and $0 < t_1 < t_2 < \dots < t_p < \tau$, $0 < \tau_1 < \tau_2, \dots < \tau_q < r < \tau$. Here, $z_t : [-r, 0] \rightarrow \mathbb{R}^n$ is defined by $z_t(\theta) = z(t + \theta)$, and η belongs to the Banach space

$$\mathcal{PW}_r = \left\{ \eta : [-r, 0] \rightarrow \mathbb{R}^n : \eta \text{ is continuous except at } s_{k\eta}, k = 1, 2, \dots, p \text{ points} \right. \\ \left. \text{where the side limits exist } \eta(s_{k\eta}^+), \eta(s_{k\eta}^-) = \eta(s_{k\eta}), \text{ and are finite} \right\}$$

with the norm

$$\|\eta\|_r = \sup_{t \in [-r, 0]} \|\eta(t)\|_{\mathbb{R}^n}.$$

There are many papers on the study of linear neutral differential equations, to mention [6, 12–14, 19, 20], particularly, the controllability of such equations has been studied in [12–14, 19, 20] where Kalman-type algebraic condition is proved (see [9]). In [6], the existence of solutions for an abstract neutral functional differential equations is discussed. To our knowledge, there are a few works on the existence of solutions for semilinear neutral equations with impulses and nonlocal conditions simultaneously. Karakostas Fixed Point Theorem will be applied to prove our main result on the existence of solutions of (1.1).

Theorem 1.1 (Karakostas Fixed Point Theorem- see[7, 10, 11]). *Let Z and Y be Banach spaces and D be a closed convex subset of Z , and let $\mathcal{B} : D \rightarrow Y$ be a continuous operator such that $\mathcal{B}(D)$ is a relatively compact subset of Y , and*

$$\mathcal{T} : D \times \overline{\mathcal{B}(D)} \rightarrow D$$

a continuous operator such that the family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive. Then, the operator equation

$$\mathcal{T}(z, \mathcal{B}(z)) = z$$

admits a solution on D .

Now, we define natural Banach spaces where the solutions of problem (1.1) will take place and present some notations to be used through this work. We begin defining the Banach spaces

$$\mathcal{PW}_{t_1 \dots t_p}([0, \tau]; \mathbb{R}^n) = \{z : [0, \tau] \rightarrow \mathbb{R}^n : z \text{ is continuous except at } t_k, k = 1, \dots, p \text{ points} \\ \text{where the side limits exist } z(t_k^+), z(t_k) = z(t_k^-), \text{ and are finite}\},$$

and

$$\mathcal{PW}_p = \left\{ \eta : [-r, \tau] \rightarrow \mathbb{R}^n : \eta \Big|_{[-r, 0]} \in \mathcal{PW}_r \text{ and } \eta \Big|_{[0, \tau]} \in \mathcal{PW}_{t_1 \dots t_p} \right\},$$

equipped with the supremum norm and

$$\|\eta\|_p = \sup_{t \in [-r, \tau]} \|\eta(t)\|_{\mathbb{R}^n},$$

respectively. We will also consider

$$\mathbb{R}^{qn} = \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \dots \times \mathbb{R}^n}_{q\text{-times}} = \prod_{k=1}^q \mathbb{R}^n$$

equipped with the norm

$$\|y\|_q = \sum_{i=1}^q \|y_i\|_{\mathbb{R}^n}.$$

Analogously, we define the Banach space

$$\mathcal{PW}_{qp} = \left\{ \eta : [-r, 0] \longrightarrow \mathbb{R}^{qn} : \eta \text{ is continuous except at } s_{k\eta}, k = 1, 2, \dots, p, \text{ points} \right. \\ \left. \text{where the side limits exist } \eta(s_{k\eta}^+), \eta(s_{k\eta}^-) = \eta(s_{k\eta}), \text{ and are finite} \right\}$$

endowed with the norm

$$\|\eta\|_{qp} = \sup_{t \in [-r, 0]} \|\eta(t)\|_q = \sup_{t \in [-r, 0]} \left(\sum_{i=1}^q \|\eta_i(t)\|_{\mathbb{R}^n} \right).$$

The functions in system (1.1) are defined as follows:

$$f_{-1}, f_1 : [0, \tau] \times \mathcal{PW}_r \longrightarrow \mathbb{R}^n, \quad h : \mathcal{PW}_{qp} \longrightarrow \mathcal{PW}_r, \quad J_k : [0, \tau] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

To conclude this section, we define the evolution operator $\mathcal{U}(t, \theta) = \Phi(t)\Phi^{-1}(\theta)$ where Φ is the fundamental matrix of the linear system of ordinary differential equations

$$y'(t) = A_0(t)y(t).$$

Also, we shall consider the following bound

$$M = \sup_{t, \theta \in [0, \tau]} \|\mathcal{U}(t, \theta)\|.$$

Remark 1.1. We will omit the subscript in the functions space norms defined above as long as this does not lead to confusion.

2. Formula for the solutions of system (1.1).

We devote this section to find a formula for solutions of the semilinear neutral differential equations with impulses and nonlocal conditions (1.1). Specifically, we transform problem (1.1) into an integral differential equation problem, which allows us to apply Karakostas Fixed Point Theorem to prove the existence of solutions for (1.1) in the next section.

Proposition 2.1. *The system (1.1) has solution z on $[-r, \tau]$ if, and only if, z is a solution of the following integral equation*

$$z(t) = \begin{cases} \mathcal{U}(t, 0) [\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ + \int_0^t \mathcal{U}(t, \theta) [A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + f_{-1}(t, z_t) \\ + \sum_{0 < t_k < t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), & t \in [0, \tau], \\ \eta(t) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(t), & t \in [-r, 0]. \end{cases} \quad (2.1)$$

Proof. (\implies) Suppose that z is a solution for system (1.1) on $[-r, \tau]$. Let

$$z_0 = \eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0).$$

• On $[0, t_1)$, z is the solution of the following system

$$\begin{cases} \frac{d}{dt} [z(t) - f_{-1}(t, z_t)] = A_0(t)z(t) + f_1(t, z_t), & t \in [0, t_1), \\ z(t) + h(z_{\tau_1}, \dots, z_{\tau_q})(t) = \eta(t), & t \in [-r, 0], \end{cases}$$

and by the variation of parameters formula

$$\begin{aligned} z(t) = & f_{-1}(t, z_t) + \mathcal{U}(t, 0)[z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ & + \int_0^t \mathcal{U}(t, \theta)[A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta, \quad t \in [0, t_1). \end{aligned}$$

As $t \rightarrow t_1^-$,

$$\begin{aligned} z(t_1^-) = & f_{-1}(t_1, z_{t_1}) + \mathcal{U}(t_1, 0)[z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ & + \int_0^{t_1} \mathcal{U}(t_1, \theta)[A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta. \end{aligned}$$

• On $[t_1, t_2)$, z is the solution of the following system

$$\begin{cases} \frac{d}{dt} [z(t) - f_{-1}(t, z_t)] = A_0(t)z(t) + f_1(t, z_t), & t \in [t_1, t_2), \\ z(t_1^+) = z(t_1) + J_1(t_1, z(t_1)) \end{cases}$$

and again the variation constant formula yields

$$\begin{aligned} z(t) = & f_{-1}(t, z_t) + \mathcal{U}(t, t_1)[z(t_1) + J_1(t_1, z(t_1)) - f_{-1}(t_1, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ & + \int_{t_1}^t \mathcal{U}(t, \theta)[A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta, \quad t \in [t_1, t_2), \end{aligned}$$

therefore

$$\begin{aligned} z(t) &= f_{-1}(t, z_t) + \mathcal{U}(t, t_1) \{ f_{-1}(t_1, z_{t_1}) + \mathcal{U}(t_1, 0) [z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + \int_0^{t_1} \mathcal{U}(t_1, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + J_1(t_1, z(t_1)) \\ &\quad - f_{-1}(t_1, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})) \} + \int_{t_1}^t \mathcal{U}(t, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta. \\ &= f_{-1}(t, z_t) + \mathcal{U}(t, t_1) \{ \mathcal{U}(t_1, 0) [z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + \int_0^{t_1} \mathcal{U}(t_1, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + J_1(t_1, z(t_1)) \} \\ &\quad + \int_{t_1}^t \mathcal{U}(t, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta. \end{aligned}$$

Using the cocycle property of \mathcal{U} ,

$$\begin{aligned} z(t) &= f_{-1}(t, z_t) + \mathcal{U}(t, 0) [z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + \int_0^{t_1} \mathcal{U}(t, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \mathcal{U}(t, t_1) J_1(t_1, z(t_1)) \\ &\quad + \int_{t_1}^t \mathcal{U}(t, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta \\ &= f_{-1}(t, z_t) + \mathcal{U}(t, 0) [z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + \int_0^t \mathcal{U}(t, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \mathcal{U}(t, t_1) J_1(t_1, z(t_1)). \end{aligned}$$

Proceeding inductively as above, we have that for $t \in [t_p, t_{p+1})$,

$$\begin{aligned} z(t) &= f_{-1}(t, z_t) + \mathcal{U}(t, 0) [z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + \int_0^t \mathcal{U}(t, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \sum_{k=1}^p \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), \quad t \in [0, \tau] \\ &= f_{-1}(t, z_t) + \mathcal{U}(t, 0) [\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + \int_0^t \mathcal{U}(t, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \sum_{0 < t_k < t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), \quad t \in [0, \tau] \end{aligned}$$

(\Leftarrow) Assume that z is solution of the integral equation (2.1).

Then, at t_1 ,

$$\begin{aligned} z(t_1^-) &= f_{-1}(t_1, z_{t_1}) + \mathcal{U}(t_1, 0)[z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + \int_0^{t_1} \mathcal{U}(t_1, \theta)[A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta, \\ z(t_1^+) &= f_{-1}(t_1, z_{t_1}) + \mathcal{U}(t_1, 0)[z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + \int_0^{t_1} \mathcal{U}(t_1, \theta)[A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \mathcal{U}(t_1, t_1)J_1(t_1, z(t_1)), \end{aligned}$$

which implies that

$$z(t_1^+) = z(t_1^-) + J_1(t_1, z(t_1)).$$

Near t_2 ,

$$\begin{aligned} z(t_2^-) &= f_{-1}(t_2, z_{t_2}) + \mathcal{U}(t_2, 0)[z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + \int_0^{t_2} \mathcal{U}(t_2, \theta)[A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \mathcal{U}(t_2, t_1)J_1(t_1, z(t_1)), \\ z(t_2^+) &= f_{-1}(t_2, z_{t_2}) + \mathcal{U}(t_2, 0)[z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + \int_0^{t_2} \mathcal{U}(t_2, \theta)[A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \mathcal{U}(t_2, t_1)J_1(t_1, z(t_1)) \\ &\quad + \mathcal{U}(t_2, t_2)J_2(t_2, z(t_2)), \end{aligned}$$

which means that

$$z(t_2^+) = z(t_2^-) + J_2(t_2, z(t_2)).$$

Proceeding inductively as above, we get that for $k = 1, 2, \dots, p$,

$$z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)).$$

On the other hand, differentiating z with respect to t , for $t \in [0, \tau)$ and $t \neq t_k, k = 1, 2, \dots, p$, we obtain that

$$\begin{aligned} \frac{d}{dt}(z(t)) &= \frac{d}{dt} \left(f_{-1}(t, z_t) + \mathcal{U}(t, 0)[z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \right. \\ &\quad \left. + \int_0^t \mathcal{U}(t, \theta)[A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \sum_{0 < t_k < t} \mathcal{U}(t, t_k)J_k(t_k, z(t_k)) \right) \\ \frac{d}{dt}(z(t)) &= \frac{d}{dt}f_{-1}(t, z_t) + A_0(t)\mathcal{U}(t, 0)[z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ &\quad + A_0(t) \int_0^t \mathcal{U}(t, \theta)[A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + A_0(t)f_{-1}(t, z_t) + f_1(t, z_t) \\ &\quad + A_0(t) \sum_{0 < t_k < t} \mathcal{U}(t, t_k)J_k(t_k, z(t_k)) \end{aligned}$$

By rearranging terms it follows that

$$\begin{aligned} \frac{d}{dt} [z(t) - f_{-1}(t, z_t)] &= A_0(t) \left\{ f_{-1}(t, z_t) + \mathcal{U}(t, 0) [z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \right. \\ &\quad + \int_0^t \mathcal{U}(t, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta \\ &\quad \left. + \sum_{0 < t_k < t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)) \right\} + f_1(t, z_t) \\ &= A_0(t) z(t) + f_1(t, z_t), \end{aligned}$$

that is to say, z is a solution of (1.1). □

3. Main Theorems

In this section we shall prove our main result about the existence of solutions for the semilinear neutral equation with impulses and nonlocal conditions (1.1) and their behavior. To achieve that, we consider the following hypotheses on the terms involving the system (1.1).

(H1) There exist constants $d_k, L_g, \gamma > 0, k = 1, 2, \dots, p$ such that $\forall y, z \in \mathbb{R}^n, t \in [0, \tau]$

i. $L_g q M < \gamma + M \sum_{k=1}^p d_k < \frac{1}{2}, \quad \|J_k(t, y) - J_k(t, z)\|_{\mathbb{R}^n} \leq d_k \|y - z\|_{\mathbb{R}^n}.$

ii. We have that $h(0) \equiv 0$ and

$$\|h(y)(t) - h(v)(t)\|_{\mathbb{R}^n} \leq L_g \sum_{i=1}^q \|y_i(t) - v_i(t)\|_{\mathbb{R}^n}, \quad y, v \in \mathcal{PW}_{qp}.$$

(H2) The function f_{-1} satisfies

i.

$$\begin{aligned} \|A_0(t) f_{-1}(t, \eta_1) - A_0(t) f_{-1}(t, \eta_2)\|_{\mathbb{R}^n} &\leq \mathcal{K} (\|\eta_1\|_r, \|\eta_2\|_r) \|\eta_1 - \eta_2\|_r, \quad \eta_1, \eta_2 \in \mathcal{PW}_r, \\ \|f_{-1}(t, \eta_1) - f_{-1}(t, \eta_2)\|_{\mathbb{R}^n} &\leq \gamma \|\eta_1 - \eta_2\|_r, \quad \eta_1, \eta_2 \in \mathcal{PW}_r \\ \|A_0(t) f_{-1}(t, \eta)\|_{\mathbb{R}^n} &\leq \Psi (\|\eta\|_r), \quad \eta \in \mathcal{PW}_r, \\ \|f_{-1}(t, \eta)\|_{\mathbb{R}^n} &\leq \Psi (\|\eta\|_r), \quad \eta \in \mathcal{PW}_r. \end{aligned}$$

and f_1 satisfies

ii.

$$\begin{aligned} \|f_1(t, \eta_1) - f_1(t, \eta_2)\|_{\mathbb{R}^n} &\leq \mathcal{K} (\|\eta_1\|_r, \|\eta_2\|_r) \|\eta_1 - \eta_2\|_r, \quad \eta_1, \eta_2 \in \mathcal{PW}_r, \\ \|f_1(t, \eta)\|_{\mathbb{R}^n} &\leq \Psi (\|\eta\|_r), \quad \eta \in \mathcal{PW}_r, \end{aligned}$$

where $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $\Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous and non decreasing functions.

(H3) There exists $\rho, \tau > 0$ such that

$$M\Psi\left(\|\eta\| + L_g q\left(\|\tilde{\eta}\| + \rho\right)\right) + \left(ML_g q + M\sum_{k=1}^p d_k\right)\left(\|\tilde{\eta}\| + \rho\right) \\ + (2M\tau + 1)\Psi\left(\|\tilde{\eta}\| + \rho\right) < \rho$$

where the function $\tilde{\eta}$ is defined as follows

$$\tilde{\eta}(t) = \begin{cases} \mathcal{U}(t, 0)\eta(0), & t \in [0, \tau], \\ \eta(t), & t \in [-r, 0]. \end{cases}$$

(H4) Assume the following relation holds

$$M\left\{L_g q(1 + \gamma) + 2\tau\mathcal{K}\left(\|\tilde{\eta}\| + \rho, \|\tilde{\eta}\| + \rho\right)\right\} < \frac{1}{2}.$$

Remark 3.1. The hypothesis **(H2)** is not a whim, it appears naturally when one studies the well-known Burgues equation and the Benjamin-Bona-Mahony equation; and since we will extend this work to infinite-dimensional Hilbert spaces, these hypotheses are considered here. For more details about it, one can see [10, 11].

Theorem 3.1. *Suppose that **(H1)**-**(H3)** hold. Then, the system (1.1) has at least one solution on $[-r, \tau]$.*

Proof. We shall transform the problem of proving the existence of solutions for system (1.1) into a fixed point problem. For this, we define the following operators

$$\mathcal{T} : \mathcal{PW}_p \times \mathcal{PW}_p \longrightarrow \mathcal{PW}_p,$$

and

$$\mathcal{B} : \mathcal{PW}_p \longrightarrow \mathcal{PW}_p$$

given by

$$\mathcal{T}(z, y)(t) = \begin{cases} y(t) + f_{-1}(t, z_t) + \sum_{0 < t_k < t} \mathcal{U}(t, t_k)J_k(t_k, z(t_k)), & t \in [0, \tau], \\ \eta(t) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(t), & t \in [-r, 0], \end{cases}$$

and

$$\mathcal{B}(y)(t) = \begin{cases} \mathcal{U}(t, 0)\left[\eta(0) - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q})(0) - f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q}))\right] \\ + \int_0^t \mathcal{U}(t, \theta)\left[A_0(\theta)f_{-1}(\theta, y_\theta) + f_1(\theta, y_\theta)\right]d\theta, & t \in [0, \tau], \\ \eta(t), & t \in [-r, 0], \end{cases}$$

respectively. We also consider the following closed and convex set

$$D = D(\rho, \tau, \eta) = \{y \in \mathcal{PW}_p : \|y - \tilde{\eta}\|_p \leq \rho\}$$

With this setting, the problem of finding solutions for system (1.1) has been reduced to the problem of finding solutions of the following operator equation

$$\mathcal{T}(z, \mathcal{B}(z)) = z.$$

The rest of the proof will be given by statements as follows:

Statement 1. \mathcal{B} is a continuous mapping.

For any $z, y \in \mathcal{PW}_p$ we have that

$$\begin{aligned} \|\mathcal{B}(z)(t) - \mathcal{B}(y)(t)\| &\leq \|\mathcal{U}(t, 0)\| \left\{ \|h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q})(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0)\| \right. \\ &\quad \left. + \|f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q})) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\| \right\} \\ &\quad + \int_0^t \|\mathcal{U}(t, \theta)\| \left\{ \|A_0(\theta)f_{-1}(\theta, z_\theta) - A_0(\theta)f_{-1}(\theta, y_\theta)\| \right. \\ &\quad \left. + \|f_1(\theta, z_\theta) - f_1(\theta, y_\theta)\| \right\} d\theta \\ &\leq M [L_g q \|z - y\| + \gamma \|g(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}) - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q})\|] \\ &\quad + M\tau [\mathcal{K}(\|z\|, \|y\|)\|z - y\| + \mathcal{K}(\|z\|, \|y\|)\|z - y\|], \\ &\leq M [L_g q \|z - y\| + \gamma L_g q \|z - y\|] \\ &\quad + 2M\tau \mathcal{K}(\|z\|, \|y\|)\|z - y\|, \end{aligned}$$

where the last two inequality comes from **(H1-ii)** and **(H2)**. It follows that

$$\|\mathcal{B}(z) - \mathcal{B}(y)\| \leq M \{L_g q (1 + \gamma) + 2\tau \mathcal{K}(\|z\|, \|y\|)\} \|z - y\|$$

by taking supremum over $t \in [-r, \tau]$. Hence \mathcal{B} is locally Lipschitz, which implies the continuity of \mathcal{B} .

Statement 2. \mathcal{B} maps bounded sets of \mathcal{PW}_p into bounded sets of \mathcal{PW}_p .

In order to prove this statement, we will show that

$$\forall R > 0 \exists \lambda > 0 \forall y \in B_R : \|\mathcal{B}(y)\| \leq \lambda,$$

where $B_R = \{z \in \mathcal{PW}_p : \|z\| \leq R\}$. Let $R > 0$ and consider $\lambda = \max\{\vartheta, \|\eta\|\}$, ϑ to be determined later. Let $y \in B_R$. Then, on one hand, we have that

$$\|\mathcal{B}(y)(t)\| = \|\eta(t)\| \leq \|\eta\|,$$

if $t \in [-r, 0]$. While, on the other hand,

$$\begin{aligned}
\|\mathcal{B}(y)(t)\| &\leq \|\mathcal{U}(t, 0)\| \|\eta(0) - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q})(0) - f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q}))\| \\
&\quad + \int_0^t \|\mathcal{U}(t, \theta)\| [\|A_0(\theta)f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\|] d\theta \\
&\leq M \{ \|\eta(0)\| + \|h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q})(0)\| + \|f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q}))\| \} \\
&\quad + \tau M [\|A_0(\theta)f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\|] \\
&\leq M \{ \|\eta(0)\| + L_g q \|y\| + \Psi(\|\eta - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q})\|) \} + \tau M 2\Psi(\|y\|) \\
&\leq M \{ \|\eta(0)\| + L_g q \|y\| + \Psi(\|\eta\| + \|h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q})\|) \} + \tau M 2\Psi(\|y\|) \\
&\leq M \{ \|\eta(0)\| + L_g q \|y\| + \Psi(\|\eta\| + L_g q \|y\|) \} + \tau M 2\Psi(\|y\|) \\
&\leq M \{ \|\eta(0)\| + L_g q R + \Psi(\|\eta\| + L_g q R) + \tau 2\Psi(R) \} = \vartheta,
\end{aligned}$$

if $t \in [0, \tau]$. Here we have used **(H1-ii)** and **(H2)**. Now, taking supremum over $t \in [-r, \tau]$, we have that

$$\|\mathcal{B}(y)\| \leq \lambda.$$

Statement 3. \mathcal{B} maps bounded sets of \mathcal{PW}_p into equicontinuous sets of \mathcal{PW}_p .

Let us consider B_R as above and let us show that $\mathcal{B}(B_R)$ is equicontinuous on $[-r, \tau]$. On $[-r, 0]$, the continuity of η immediately implies the result. On $(0, \tau]$, we have that

$$\begin{aligned}
\|\mathcal{B}(y)(t_2) - \mathcal{B}(y)(t_1)\| &\leq \|\mathcal{U}(t_2, 0) - \mathcal{U}(t_1, 0)\| \|\eta(0) - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q})(0) \\
&\quad - f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q}))\| \\
&\quad + \int_0^{t_1} \|\mathcal{U}(t_2, \theta) - \mathcal{U}(t_1, \theta)\| [\|A_0(\theta)f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\|] d\theta \\
&\quad + \int_{t_1}^{t_2} \|\mathcal{U}(t_2, \theta)\| [\|A_0(\theta)f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\|] d\theta \\
&\leq \|\mathcal{U}(t_2, 0) - \mathcal{U}(t_1, 0)\| \left\{ \|\eta(0)\| + L_g q \|y\| \right. \\
&\quad \left. + \|f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \dots, y_{\tau_q}))\| \right\} \\
&\quad + \int_0^{t_1} \|\mathcal{U}(t_2, \theta) - \mathcal{U}(t_1, \theta)\| [\|A_0(\theta)f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\|] d\theta \\
&\quad + \int_{t_1}^{t_2} \|\mathcal{U}(t_2, \theta)\| [\|A_0(\theta)f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\|] d\theta \\
&\leq \|\mathcal{U}(t_2, 0) - \mathcal{U}(t_1, 0)\| \left\{ \|\eta(0)\| + L_g q \|y\| + \Psi(\|\eta\| + L_g q \|y\|) \right\} \\
&\quad + 2\Psi(\|y\|) \int_0^{t_1} \|\mathcal{U}(t_2, \theta) - \mathcal{U}(t_1, \theta)\| d\theta + 2M\Psi(\|y\|)(t_2 - t_1)
\end{aligned}$$

$$\begin{aligned} &\leq \|\mathcal{U}(t_2, 0) - \mathcal{U}(t_1, 0)\| \left\{ \|\eta(0)\| + L_g q R + \Psi (\|\eta\| + L_g q R) \right\} \\ &\quad + 2\Psi (R) \int_0^{t_1} \|\mathcal{U}(t_2, \theta) - \mathcal{U}(t_1, \theta)\| d\theta + 2M\Psi (R) (t_2 - t_1) \rightarrow 0 \end{aligned}$$

as $t_2 \rightarrow t_1$ by the continuity of \mathcal{U} and the fact that $\|\eta(0)\| + L_g q R + \Psi (\|\eta\| + L_g q R)$ is bounded. Here we have considered **(H1-ii)** and **(H2)**. This shows that $\mathcal{B}(B_R)$ is equicontinuous.

Statement 4. *The subset $\mathcal{B}(D)$ is relatively compact in \mathcal{PW}_p .*

Let us prove Statement 4. Let D be a bounded subset of \mathcal{PW}_p . By Statements 2 and 3, $\mathcal{B}(D)$ is bounded and equicontinuous in \mathcal{PW}_p . Let $\{y_n\}_{n \in \mathbb{N}} \subseteq \mathcal{B}(D)$, then

$$y_n \Big|_{[-r, 0]} = \eta, \forall n \in \mathbb{N}.$$

Hence, $y_n \Big|_{[-r, 0]}$ converges uniformly on $[-r, 0]$.

Now, putting $\varphi_n = y_n \Big|_{[0, \tau]}$, we get that $\{\varphi_n\}_{n \in \mathbb{N}} \subseteq \mathcal{PW}_{t_1 \dots t_p}$.

Let us put $t_0 = 0$ and $t_{p+1} = \tau$. Then, applying Arzela-Ascoli Theorem, the sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ contains a subsequence $\{\varphi_n^1\}_{n \in \mathbb{N}}$ that converges in the interval $[t_0, t_1]$. Now, applying Arzela-Ascoli Theorem again, we get that the sequence $\{\varphi_n^1\}_{n \in \mathbb{N}}$ contains a subsequence $\{\varphi_n^2\}_{n \in \mathbb{N}}$ that converges in the interval $[t_1, t_2]$. Continuing with this process we find a subsequence $\{\varphi_n^{p+1}\}_{n \in \mathbb{N}}$ of $\{\varphi_n\}_{n \in \mathbb{N}}$ that converges in each interval $[t_k, t_{k+1}]$, with $k = 0, 1, 2, \dots, p$. Therefore,

$$\varphi_n^{p+1} = y_n^{p+1} \Big|_{[0, \tau]} \text{ converges on } [0, \tau].$$

Consequently, $\{\varphi_n^{p+1}\}_{n \in \mathbb{N}} = \{y_n^{p+1}\}_{n \in \mathbb{N}}$ converges uniformly on $[-r, \tau]$. Thus, $\mathcal{B}(D)$ is relatively compact, and the proof of Statement 4 is completed.

Statement 5. *The family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive.*

On the one hand, for any $u, v \in \mathcal{PW}_p$ and $t \in [-r, 0]$, we get that

$$\begin{aligned} \|\mathcal{T}(u, \mathcal{B}(y))(t) - \mathcal{T}(v, \mathcal{B}(y))(t)\| &\leq \|h(u_{\tau_1}, u_{\tau_2}, \dots, u_{\tau_q})(t) - h(v_{\tau_1}, v_{\tau_2}, \dots, v_{\tau_q})(t)\| \\ &\leq L_g q \|u - v\| \\ &\leq M L_g q \|u - v\|. \end{aligned}$$

While on the other hand, by using **(H1-i)** and **(H2-i)**, for all $t \in (0, \tau]$ we obtain

that

$$\begin{aligned}
\|\mathcal{T}(u, \mathcal{B}(y))(t) - \mathcal{T}(v, \mathcal{B}(y))(t)\| &\leq \|f_{-1}(t, u_t) - f_{-1}(t, v_t)\| \\
&\quad + \sum_{0 < t_k < t} \|\mathcal{U}(t, t_k) [J_k(t_k, u(t_k)) - J_k(t_k, v(t_k))]\| \\
&\leq \gamma \|u - v\| + M \sum_{k=1}^p \|J_k(t_k, u(t_k)) - J_k(t_k, v(t_k))\| \\
&\leq \gamma \|u - v\| + M \sum_{k=1}^p d_k \|u(t_k) - v(t_k)\| \\
&\leq \gamma \|u - v\| + M \|u - v\| \sum_{k=1}^p d_k \\
&\leq \left(\gamma + M \sum_{k=1}^p d_k \right) \|u - v\|.
\end{aligned}$$

It follows that

$$\|\mathcal{T}(u, \mathcal{B}(y)) - \mathcal{T}(v, \mathcal{B}(y))\| \leq \left(\gamma + M \sum_{k=1}^p d_k \right) \|u - v\| \leq \frac{1}{2} \|u - v\|$$

by taking supremum over $t \in [-r, \tau]$ and using **(H1-i)**. This shows that $\mathcal{T}(\cdot, \mathcal{B}(y))$ is a contraction which does not depend on $y \in \mathcal{B}(D)$.

Statement 6. *The inclusion $\mathcal{T}(\cdot, \mathcal{B}(\cdot))(D(\rho, \tau, \eta)) \subset D(\rho, \tau, \eta)$ holds.*

Let $z \in D(\rho, \tau, \eta)$ be arbitrary. Notice that

$$\mathcal{T}(z, \mathcal{B}(z))(t) = \begin{cases} \mathcal{U}(t, 0) [\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))] \\ \quad + \int_0^t \mathcal{U}(t, \theta) [A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + f_{-1}(t, z_t) \\ \quad + \sum_{0 < t_k < t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), & t \in [0, \tau], \\ \eta(t) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(t), & t \in [-r, 0]. \end{cases}$$

On the one hand, for $t \in [-r, 0]$, we have that

$$\begin{aligned}
\|\mathcal{T}(z, \mathcal{B}(z))(t) - \tilde{\eta}(t)\| &\leq \|g(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(t)\| \\
&\leq L_g q \|z\| \\
&\leq M L_g q \|z\| \\
&\leq M L_g q (\|\tilde{\eta}\| + \rho) \\
&< \rho.
\end{aligned}$$

While on the other hand, for $t \in [0, \tau]$, we have that

$$\begin{aligned} \|\mathcal{T}(z, \mathcal{B}(z))(t) - \tilde{\eta}(t)\| &\leq M \|h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\| \\ &\quad + \int_0^t \|\mathcal{U}(t, \theta) [A_0(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]\| d\theta + \|f_{-1}(t, z_t)\| \\ &\quad + \sum_{0 < t_k < t} \|\mathcal{U}(t, t_k)J_k(t_k, z(t_k))\| \\ &\leq M \{L_g q \|z\| + \|f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\|\} \\ &\quad + 2M\tau\Psi(\|z\|) + \Psi(\|z\|) + M \sum_{0 < t_k < t} \|J_k(t_k, z(t_k))\| \\ &\leq M \{L_g q \|z\| + \Psi(\|\eta\| + L_g q \|z\|)\} \\ &\quad + 2M\tau\Psi(\|z\|) + \Psi(\|z\|) + \left(M \sum_{k=1}^p d_k\right) \|z\| \\ &\leq M \{L_g q (\|\tilde{\eta}\| + \rho) + \Psi(\|\tilde{\eta}\| + L_g q (\|\tilde{\eta}\| + \rho))\} \\ &\quad + 2M\tau\Psi(\|\tilde{\eta}\| + \rho) + \Psi(\|\tilde{\eta}\| + \rho) + \left(M \sum_{k=1}^p d_k\right) (\|\tilde{\eta}\| + \rho) \\ &\leq M\Psi(\|\tilde{\eta}\| + L_g q (\|\tilde{\eta}\| + \rho)) + \left(ML_g q + M \sum_{k=1}^p d_k\right) (\|\tilde{\eta}\| + \rho) \\ &\quad + (2M\tau + 1)\Psi(\|\tilde{\eta}\| + \rho) < \rho. \end{aligned}$$

Here we have used **(H3)**. Now, by taking supremum over $t \in [-r, \tau]$, we get that

$$\|\mathcal{T}(z, \mathcal{B}(z)) - \tilde{\eta}\| \leq \rho.$$

and by Karakostas Fixed Point Theorem the operator equation

$$\mathcal{T}(z, \mathcal{B}(z)) = z$$

admits a solution on D . This finishes the proof. □

Theorem 3.2. *System (1.1) has a unique solution if **(H4)** is additionally assumed.*

Proof. Suppose u and v are two solutions of system (1.1). Now, considering **(H1)** and **(H2)** we have that

$$\begin{aligned} \|u(t) - v(t)\| &\leq \|\mathcal{U}(t, 0)\| \left\{ \|h(u_{\tau_1}, u_{\tau_2}, \dots, u_{\tau_q})(0) - h(v_{\tau_1}, v_{\tau_2}, \dots, v_{\tau_q})(0)\| \right. \\ &\quad \left. + \|f_{-1}(0, \eta - h(u_{\tau_1}, u_{\tau_2}, \dots, u_{\tau_q})) - f_{-1}(0, \eta - h(v_{\tau_1}, v_{\tau_2}, \dots, v_{\tau_q}))\| \right\} \\ &\quad + \int_0^t \|\mathcal{U}(t, \theta)\| \left\{ \|A_0(\theta)f_{-1}(\theta, u_\theta) - A_0(\theta)f_{-1}(\theta, v_\theta)\| + \|f_1(\theta, u_\theta) - f_1(\theta, v_\theta)\| \right\} d\theta \end{aligned}$$

$$\begin{aligned}
& + \|f_{-1}(t, u_t) - f_{-1}(t, v_t)\| + \sum_{0 < t_k < t} \|\mathcal{U}(t, t_k)\| \|J_k(t_k, u(t_k)) - J_k(t_k, v(t_k))\| \\
& \leq M \{L_{gq}(1 + \gamma) + 2\tau\mathcal{K}(\|u\|, \|v\|)\} \|u - v\| + \left(\gamma + M \sum_{k=1}^p d_k\right) \|u - v\| \\
& \leq M \left\{L_{gq}(1 + \gamma) + 2\tau\mathcal{K}(\|\tilde{\eta}\| + \rho, \|\tilde{\eta}\| + \rho)\right\} \|u - v\| + \frac{1}{2}\|u - v\|
\end{aligned}$$

Bearing in mind the hypothesis **(H4)**, and taking supremum over $t \in [-r, \tau]$, we have that

$$\|u - v\| \leq \omega \|u - v\|$$

with $0 \leq \omega < 1$. This implies $\|u - v\| = 0$, and therefore $u = v$. \square

Next, we consider the following subset \tilde{D} of \mathbb{R}^n :

$$\tilde{D} = \{v \in \mathbb{R}^n : \|v\|_{\mathbb{R}^n} \leq \rho\}. \quad (3.1)$$

Therefore, for all $y \in D$ we have $y(t) - \tilde{\eta}(t) \in \tilde{D}$ for $t \in [-r, \tau]$.

Definition 3.1. We shall say that $[-r, \theta_1)$ is a maximal interval of existence for the solution z of problem (1.1) if there is not solution of (1.1) on $[-r, \theta_2)$ with $\theta_2 > \theta_1$.

Theorem 3.3. *Suppose that the conditions of Theorem 3.1 hold. If z is a solution of problem (1.1) on $[-r, \theta_1)$ and θ_1 is maximal, then either $\theta_1 = +\infty$ or there exists a sequence $\tau_n \rightarrow \theta_1$ as $n \rightarrow \infty$ such that $z(\tau_n) - \tilde{\eta}(\tau_n) \rightarrow \partial\tilde{D}$.*

Proof. Suppose $\theta_1 < \infty$. For the purpose of contradiction assume the existence of a neighborhood N of $\partial\tilde{D}$ such that $\{z(t) - \tilde{\eta}(t)\}$ does not enter in it, for $0 < \theta_2 \leq t < \theta_1$. We can take $N = \tilde{D} \setminus B$, where B is a closed subset of \tilde{D} , then $z(t) - \tilde{\eta}(t) \in B$ for $0 < t_p < \theta_2 \leq t < \theta_1$. We need to prove that $\lim_{t \rightarrow \theta_1^-} \{z(t) - \tilde{\eta}(t)\} = z_1 - \tilde{\eta}(\theta_1) \in B$, which is enough to prove that $\lim_{t \rightarrow \theta_1^-} z(t) = z_1$. Indeed, if we consider $0 < t_p < \theta_2 \leq \ell < t < \theta_1$,

then:

$$\begin{aligned}
\|z(t) - z(\ell)\| & \leq \|\mathcal{U}(t, 0) - \mathcal{U}(\ell, 0)\| (\|\eta(0)\| + \|h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0)\|) \\
& \quad + \|\|f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\|\| \\
& \quad + \int_0^\ell \|\mathcal{U}(t, \theta) - \mathcal{U}(\ell, \theta)\| \|A(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)\| d\theta \\
& \quad + \int_\ell^t \|\mathcal{U}(t, \theta)\| \|A(\theta)f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)\| d\theta + \|f_{-1}(t, z_t) - f_{-1}(\ell, z_\ell)\| \\
& \quad + \sum_{0 < t_k < \ell} \|\mathcal{U}(t, t_k) - \mathcal{U}(\ell, t_k)\| \|J_k(t_k, z(t_k))\| \\
& \quad + \sum_{\ell < t_k < t} \|\mathcal{U}(t, t_k)\| \|J_k(t_k, z(t_k))\|
\end{aligned}$$

$$\begin{aligned}
 &\leq \|\mathcal{U}(t, 0) - \mathcal{U}(\ell, 0)\| (\|\eta(0)\| + L_g q \|z\| + \Psi(\|\eta\| + L_g q \|z\|)) \\
 &\quad + \left(\int_0^\ell \|\mathcal{U}(t, \theta) - \mathcal{U}(\ell, \theta)\| d\theta + \int_\ell^t \|\mathcal{U}(t, \theta)\| d\theta \right) 2\Psi(\|z\|) \\
 &\quad + \|f_{-1}(t, z_t) - f_{-1}(\ell, z_\ell)\| + \|\mathcal{U}(t, \ell) - I\| \sum_{k=1}^q \|\mathcal{U}(\ell, t_k)\| \|J_k(z(t_k))\| \\
 &\quad + \sum_{\ell < t_k < t} \|\mathcal{U}(t, t_k)\| \|J_k(t_k, z(t_k))\| \\
 &\leq \|\mathcal{U}(t, 0) - \mathcal{U}(\ell, 0)\| (\|\eta(0)\| + L_g q \|z\| + \Psi(\|\eta\| + L_g q \|z\|)) \\
 &\quad + \left(\int_0^\ell \|\mathcal{U}(t, \theta) - \mathcal{U}(\ell, \theta)\| d\theta + \int_\ell^t \|\mathcal{U}(t, \theta)\| d\theta \right) 2\Psi(\|z\|) \\
 &\quad + \|f_{-1}(t, z_t) - f_{-1}(\ell, z_\ell)\| + \|\mathcal{U}(t, \ell) - I\| M \sum_{k=1}^q \|J_k(z(t_k))\| \\
 &\quad + \sum_{\ell < t_k < t} \|\mathcal{U}(t, t_k)\| \|J_k(t_k, z(t_k))\|
 \end{aligned}$$

Since \mathcal{U} is uniformly continuous for $t \geq 0$, then $\|z(t) - z(l)\|_{\mathbb{R}^n}$ goes to zero as $l \rightarrow \theta_1^-$. Therefore, $\lim_{t \rightarrow \theta_1^-} z(t) = z_1$ exists in \mathbb{R}^n and, since B is closed, $z_1 - \tilde{\eta}(\theta_1)$ belongs to B . This will contradict the maximality of θ_1 . In fact, we have that $z_1 \in B + \tilde{\eta}(\theta_1)$ is contained in the interior of the ball $\tilde{D} + \tilde{\eta}(\theta_1)$. Hence, $z(\cdot)$ can be extended to $[-r, \theta_1]$. In this regard, for ϵ small enough, the following initial value problem admit only one solutions on $[-r, \theta_1 + \epsilon)$

$$\begin{cases} \frac{d}{dt}[u(t) - f_{-1}(t, u_t)] = A_0(t)u(t) + f_1(t, u_t), & t \in [\theta_1, \theta_1 + \epsilon), \\ u(\theta) = z(\theta), & \theta \in [\theta_1 - r, \theta_1]. \end{cases} \tag{3.2}$$

This is a contradiction with the maximality of θ_1 . So, the proof is completed. \square

Corollary 3.1. *In the conditions of Theorem 3.1, if the second part of hypothesis (H1) is changed to*

$$\|f_1(t, \eta)\| \leq \mu(t)(1 + \|\eta(0)\|_{\mathbb{R}^n}), \quad \eta \in \mathcal{PW}_r, \quad t \in [-r, \infty),$$

where μ is a continuous function on $[-r, \infty)$, then a unique solution of problem (1.1) exists on $[-r, \infty)$.

Proof.

$$\begin{aligned}
 \|z(t)\| &\leq \|\mathcal{U}(t, 0)\| \|\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(0) - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\| \\
 &\quad + M \int_0^t \|A_0(\theta) f_{-1}(\theta, z_\theta)\| + \|f_1(\theta, z_\theta)\| d\theta + \|f_{-1}(t, z_t)\| \\
 &\quad + \sum_{0 < t_k < t} M \|J_k(t_k, z(t_k))\|
 \end{aligned}$$

$$\begin{aligned} &\leq \|\mathcal{U}(t, 0)\| (\|\eta(0)\| + L_g q \|z\| + \Psi (\|\eta\| + L_g q \|z\|)) \\ &\quad + M \int_0^t \|A_0(\theta) f_{-1}(\theta, z_\theta)\| + \mu(\theta)(1 + \|z(\theta)\|) d\theta + \|f_{-1}(t, z_t)\| \\ &\quad + \sum_{0 < t_k < t} M d_k \|z(t_k)\| \end{aligned}$$

Then, applying Gronwall Inequality for impulsive differential equations (see [8, 15, 16, 18]), we obtain that

$$\|z(t)\|_{\mathbb{R}^n} \leq M \left(\|z(0)\|_{\mathbb{R}^n} + \int_0^\tau \mu(\theta) d\theta \right) \prod_{t_0 < t_k < t} (1 + M d_k) e^{\int_0^\tau M \mu(\theta) d\theta},$$

This implies that $\|z(t)\|_{\mathbb{R}^n}$ remains bounded as $t \rightarrow \theta_1$ and applying Theorem 3.3 we get the result. □

4. Global Lipschitz Conditions

This section will assume stronger hypotheses on the nonlinear terms that allow us to apply Banach Fixed Point Theorem. Specifically, we will suppose that the nonlinear functions that appear in our system are globally Lipschitz. Moreover, we shall consider the following simpler system

$$\begin{cases} \frac{d}{dt}[z(t) - f(t, z_t)] = A_0(t)z(t) + F(t, z_t), & t \in [0, \tau] \setminus \{t_1, t_2, \dots, t_p\} \\ z(s) = g(z)(s) + \phi(s), & s \in [-r, 0] \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), & k = 1, 2, \dots, p, \end{cases} \tag{4.1}$$

where the nonlocal condition $z(s) = g(z)(s) + \phi(s), \quad s \in [-r, 0]$ means

$$z(s) = g \left(z \Big|_{[-r, 0]} \right) (s) + \phi(s), \quad s \in [-r, 0].$$

The functions $f, F : [0, \tau] \times \mathcal{PW}_r \rightarrow \mathbb{R}^n$ are smooth enough satisfying certain conditions that will be specified later, and $J_k : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n, k = 1, 2, \dots, p$, are continuous and represents the impulsive effect in the system (4.1), the continuous function $g : \mathcal{PW}_r \rightarrow \mathcal{PW}_r$ represent the nonlocal conditions, this function acts as a feedback operator which adjusts a part of the past when the initial function is present, or even, the whole past when the function ϕ is absent according to some precise future requirements (see [1]). The advantage of using nonlocal conditions is that measurements at more places can be incorporated to get better models. For more details and physical interpretations about nonlocal condition see [1–5, 21] and references therein.

Now, assuming a global Lipschitz condition, we will prove that system (4.1) admits a unique solution defined on $[0, \tau]$ by applying Banach Fixed Point Theorem. In this regards, we suppose the following global Lipschitz condition on the nonlinear terms:

(L1) There exist positive constants L_f and L_F such that for all $t \in [0, \tau]$, $\phi, \tilde{\phi} \in \mathcal{PW}_r$

$$\begin{aligned} \|f(t, \phi) - f(t, \tilde{\phi})\| &\leq L_f \|\phi - \tilde{\phi}\|_r, \\ \|F(t, \phi) - F(t, \tilde{\phi})\| &\leq L_F \|\phi - \tilde{\phi}\|_r. \end{aligned}$$

(L2) There exist nonnegative constants $d_k, k = 1, 2, \dots, p$ such that for all $t \in [0, \infty)$, $z, \tilde{z} \in \mathbb{R}^n$

$$\|J_k(t, z) - J_k(t, \tilde{z})\|_{\mathbb{R}^n} \leq d_k \|z - \tilde{z}\|_{\mathbb{R}^n}.$$

(L3) There exists a nonnegative constant L_g such that for all $\phi, \psi \in \mathcal{PW}_r$

$$\|g(\phi) - g(\psi)\|_r \leq L_g \|\phi - \psi\|_r.$$

(L4)

$$L_f + M[L_g + L_f L_g + \|A_0\|L_f \tau + L_F \tau + \sum_{k=1}^p d_k] < 1,$$

where $\|A_0\| = \max\{\|A_0(t)\| : t \in [0, \tau]\}$.

Proposition 4.1. *Let $\phi \in \mathcal{PW}_r$. Then z is solution of system (4.1) if and only if z satisfies the integral equation*

$$z(t) = \begin{cases} g(z)(t) + \phi(t), & t \in [-r, 0], \\ f(t, z_t) + \mathcal{U}(t, 0)[g(z)(0) + \phi(0) - f(0, g(z)(0) + \phi(0))] \\ + \int_0^t \mathcal{U}(t, s)A_0(s)f(s, z_s)ds \\ + \int_0^t \mathcal{U}(t, s)F(s, z_s)ds + \sum_{0 < t_k < t} \mathcal{U}(t, t_k)J_k(t_k, z(t_k)), & t \in [0, \tau], \end{cases} \quad (4.2)$$

Theorem 4.1. *Suppose that (L1)-(L4) hold. Then for $\phi \in \mathcal{PW}_r$ the system (4.1) has a unique solution defined on $[0, \tau]$.*

Proof. We shall apply Banach Contraction Mapping Theorem, in this regard, we will define the following operator $\mathcal{T} : \mathcal{PW}_p \rightarrow \mathcal{PW}_p$ by

$$\mathcal{T}(t) = \begin{cases} g(z)(t) + \phi(t), & t \in [-r, 0], \\ f(t, z_t) + \mathcal{U}(t, 0)[g(z)(0) + \phi(0) - f(0, g(z)(0) + \phi(0))] \\ + \int_0^t \mathcal{U}(t, s)A_0(s)f(s, z_s)ds \\ + \int_0^t \mathcal{U}(t, s)F(s, z_s)ds + \sum_{0 < t_k < t} \mathcal{U}(t, t_k)J_k(t_k, z(t_k)), & t \in [0, \tau]. \end{cases} \quad (4.3)$$

If $t \in [-r, 0]$, then

$$\begin{aligned} \|(\mathcal{T}z)(t) - (\mathcal{T}\tilde{z})(t)\| &= \|g(z)(t) - g(\tilde{z})(t)\| \leq \|g(z) - g(\tilde{z})\|_{[-r, 0]} \|p \\ &\leq L_g \|(z - \tilde{z})\|_{[-r, 0]} \|p_{\mathcal{PW}_r} \leq L_g \|z - \tilde{z}\|_p. \end{aligned}$$

If $t \in [0, \tau]$, then

$$\begin{aligned}
\|(\mathcal{T}z)(t) - (\mathcal{T}\tilde{z})(t)\| &\leq \|f(t, z_t) - f(t, \tilde{z}_t)\| + \|\mathcal{U}(t, 0)\| [\|g(z)(0) - g(\tilde{z})(0)\| \\
&\quad + \|f(0, g(z)(0) + \phi(0)) - f(0, g(\tilde{z})(0) + \phi(0))\|] \\
&\quad + \int_0^t \|\mathcal{U}(t, s)\| \|A_0(s)\| \|f(s, z_s) - f(s, \tilde{z}_s)\| ds \\
&\quad + \int_0^t \|\mathcal{U}(t, s)\| \|F(s, z_s) - F(s, \tilde{z}_s)\| ds \\
&\quad + \sum_{0 < t_k < t} \|\mathcal{U}(t, t_k)\| \|J_k(t_k, z(t_k)) - J_k(t_k, \tilde{z}(t_k))\| \\
&\leq L_f \|z_t - \tilde{z}_t(t)\| + M [L_g \|z - \tilde{z}\|_{\mathcal{PW}_p} + L_f \|g_\tau(z)(0) - g_\tau(\tilde{z}(0))\|] \\
&\quad + M \|A_0\| L_f \int_0^t \|z_s - \tilde{z}_s\| ds + M L_F \int_0^t \|z_s - \tilde{z}_s\| ds \\
&\quad + M \sum_{0 < t_k < t} d_k \|z(t_k) - \tilde{z}(t_k)\| \\
&\leq \left(L_f + M \left[L_g + L_f L_g + \|A_0\| L_f \tau + L_F \tau + \sum_{k=1}^p d_k \right] \right) \|z - \tilde{z}\|_p.
\end{aligned}$$

Thus,

$$\|\mathcal{T}z - \mathcal{T}\tilde{z}\|_p \leq \left(L_f + M \left[L_g + L_f L_g + \|A_0\| L_f \tau + L_F \tau + \sum_{k=1}^p d_k \right] \right) \|z - \tilde{z}\|_p,$$

so, the operator \mathcal{T} satisfies all the assumptions of the Banach Contraction Mapping Theorem, and therefore \mathcal{T} has only one fixed point in the space \mathcal{PW}_τ , which is the solution of problem (4.1). This completes the proof. \square

5. Example

In this section, we consider an example of semilinear neutral differential equations with impulses, delay and nonlocal conditions such that Theorem 4.1 can be applied. Let us consider the following system

$$\begin{cases} \frac{d}{dt} \left[z(t) - \left(1 + \frac{\tan z(t-2)}{8(t+10)^2} \right) \right] = z(t) + e^{-\frac{z(t-2)}{10(t+5)^3}}, & t \in [0, \tau] \\ z(s) = \left(1 + \frac{\sin z}{30^2} \right) (s) + \phi(s), & s \in [-2, 0] \\ z(t_k^+) = z(t_k^-) + 1 + \frac{\cos(z(t_k^-))}{4(t_k+8)^4}, & k = 1, 2. \end{cases} \quad (5.1)$$

Here $t_1 = \frac{5}{2}$, $t_2 = \frac{9}{2}$ and $\tau = 5$. In this example, the terms related to system (4.1) are given by: $f(t, z) = 1 + \frac{\tan(z)}{8(t+10)^2}$, $F(t, z) = e^{-\frac{z}{10(t+5)^3}}$, $g(z) = 1 + \frac{\sin(z)}{30^2}$, $J_k(t, z) = 1 + \frac{\cos(z)}{4(t+8)^4}$ and $A_0(t) = 1$. Then we have,

$$\begin{aligned}
 |f(t, z) - f(t, \tilde{z})| &= \frac{1}{8(t+10)^2} |\tan(z) - \tan(\tilde{z})| \leq \frac{1}{8 \cdot 10^2} |z - \tilde{z}|_r, \\
 |F(t, z) - F(t, \tilde{z})| &= \left| e^{-\frac{1}{10(t+5)^3} z} - e^{-\frac{1}{10(t+5)^3} \tilde{z}} \right| \leq \frac{1}{10 \cdot 5^3} |z - \tilde{z}|_r, \\
 |J_k(t, z) - J_k(t, \tilde{z})| &= \frac{1}{4(t+8)^4} |\cos(z) - \cos(\tilde{z})| \leq \frac{1}{4 \cdot 8^4} |z - \tilde{z}|_r, \\
 |g(z) - g(\tilde{z})|_r &= \frac{1}{30^2} |\sin(z) - \sin(\tilde{z})|_{\mathcal{PW}_r} \leq \frac{1}{30^2} |z - \tilde{z}|_r,
 \end{aligned}$$

and

$$L_f + M[L_g + L_f L_g + |A_0|L_f \tau + L_F \tau + d_1 + d_2] \leq 0.63.$$

Hence, the conditions (L1)-(L4) are satisfied. Consequently, Theorem 4.1 ensures the existence of solutions for problem (5.1).

6. Final Remark

In this paper, first of all, we have proved the existence, uniqueness, and the globally defined solutions of a semilinear neutral differential equation with impulses and non-local conditions assuming that the nonlinear terms are locally Lipschitz. After that, we assume that the nonlinear functions that involve system (4.1) are globally Lipschitz, which allows us to prove the existence and uniqueness of solutions by applying Banach Fixed Point Theorem. Finally, we believe that this work can be extended to infinite dimension systems in Hilbert spaces, where the operator A_0 is no longer a matrix, instead, it will be the infinitesimal generator of a strongly continuous compact semigroup, and $-A_0$ a sectorial operator. In this way, the fractional powered spaces can be defined, allowing us to admit nonlinear terms involving spatial derivatives, like in the following neutral partial differential equations of Burges equation type:

$$\begin{cases}
 \frac{\partial}{\partial t} \left[z(t, x) + \int_0^t \int_0^\pi b(\theta - t, y, x) z(\theta, y) dy d\theta \right] = \nu z_{xx}(t, x) - z(t - r) z_x(t - r) \\
 \quad + f(t, z(t - r, x)), \quad t \neq t_k, \\
 z(t, 0) = z(t, 1) = 0, \quad t \in [0, \tau] \\
 z(\theta, x) + h(z(\tau_1 + \theta, x), \dots, z(\tau_q + \theta, x)) = \eta(\theta, x), \quad x \in [0, 1], \\
 z(t_k^+, x) = z(t_k^-, x) + J_k(z(t_k, x)), \quad x \in \Omega, \quad k = 1, 2, 3, \dots, p,
 \end{cases} \tag{6.1}$$

where $\eta \in \mathcal{PW}_{1/2}(-r, 0; H_0^1) = \mathcal{PW}_{1/2}(-r, 0; Z^{1/2})$, with $Z = L_2[0, 1]$, $Z^{1/2} = D((-\Delta)^{1/2})$ and the functions f, J_k, h are locally Lipschitz.

Acknowledgment

The authors would like to express their thanks to the editor and anonymous referees for constructive comments and suggestions that improved the quality of this manuscript.

This work has been supported by Yachay Tech and ULA

References

- [1] M. Burlica, M. Necula, D. Rosu, I. Vrabie, *Delay Differential Evolutions Subjected to Nonlocal Initial Conditions*, Monographs and Research Notes in Mathematics, Taylor and Francis Group, LLC, 2016.
- [2] L. Byszewski, *Existence and uniqueness of solutions of nonlocal problems for hiperbolic equation $u_{xt} = F(x, t, u, u_x)$* , J. of App. Math. and Stoch. Anal. 3 (1990) 163–168.
- [3] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. of Math. Anal. and Appl. 162 (1991) 494–505.
- [4] L. Byszewski, V. Lakshmikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal. 40 (1990) 11–19.
- [5] J. Chabrowski, *On nonlocal problems for parabolic equations*, Nagoya Math. J. 93 (1984) 109–131.
- [6] E. Hernández, M. Pierri, *On abstract neutral differential equations with state-dependent delay*, Journal of Fixed Point Theory and Applications 20 (2018) 46–48.
- [7] G.L. Karakostas, *An extension of Krasnosel'skii's fixed point theorem for contraction and compact mappings*, Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center 22 (2003) 181–191.
- [8] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
- [9] E.B. Lee, L. Markus, *Foundations of Optimal Control Theory*, Wiley, New York, 1967.
- [10] H. Leiva, P. Sundar, *Existence of solutions for a class of semilinear evolution equations whit impulses and delays*, Journal of Nonlinear Evolution Equations and Applications 2017 (2017) 95–108.
- [11] H. Leiva, *Karakostas's fixed point theorem and the existence of solutions for impulsive semilinear evolution equations with delay and nonlocal conditions*, Communications in Mathematical Analysis 21 (2) (2018) 68–91.
- [12] A. Manitius, *Necessary and sufficient conditions of approximate controllability for general linear retarded systems*, SIAM Journal on Control and Optimization 19 (1981) 516–532.
- [13] R. Rabah, G.M. Sklyar. *Exact controllability of linear neutral type systems by the moment problem approach*, Proceedings of the IEEE Conference on Decision and Control, 2008, 2734–2739.

- [14] H. Reinbacher, *New algebraic conditions for controllability of neutral differential equations*, Journal of Optimization Theory and Applications 54 (1987) 93–111.
- [15] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific Series on Nonlinear Science Series A, Vol. 14, 1995.
- [16] S. Selvi, M. Mallika Arjunan, *Controllability results for impulsive differential systems with finite delay*, Journal Nonlinear Sciences and Applications 5 (2012) 206–219.
- [17] A.S.C. Sinha, C.F. Yokomoto, *Null controllability of a nonlinear system with variable time delay*, IEEE Trans. Auto. Cont. AC-25 (1980) 1234–1236.
- [18] R. Shikharchand Jain, M. Baburao Dhage, *On mild solutions of nonlocal semilinear impulsive functional integro-differential equations*, Applied Mathematics E-Notes 13 (2014) 109–119.
- [19] D.A. O’Connor, T.J. Tarn. *On the function space controllability of linear neutral systems*, SIAM Journal on Control and Optimization 21 (1983) 306–328.
- [20] G.R. Underwood, E.N. Chukwu, *Null controllability of nonlinear neutral differential equations*, Journal of Mathematical Analysis and Applications 129 (1982) 326–345.
- [21] I.I. Vrabie, *Delay evolution equations with mixed nonlocal plus local initial conditions*, Comm. in Cont. Math. 17 (2) (2015) 1350035.

DOI: 10.7862/rf.2022.5

Hugo Leiva

email: hleiva@yachaytech.edu.ec, hleiva@ula.ve
ORCID: 0000-0002-3521-6253
Department of Mathematics
School of Mathematical and Computational Sciences
Yachay Tech
San Miguel de Urcuqui
Imbabura
ECUADOR

Lenin Riera

email: lenin.riera@yachaytech.edu.ec
ORCID: 0000-0002-4803-4918
Department of Mathematics
School of Mathematical and Computational Sciences
Yachay Tech
San Miguel de Urcuqui

Imbabura
ECUADOR

Sebastián Lalvay

email: sebastian.lalvay@yachaytech.edu.ec

ORCID: 0000-0002-0480-0828

Department of Mathematics

School of Mathematical and Computational Sciences

Yachay Tech

San Miguel de Urcuqui

Imbabura

ECUADOR

Received 14.02.2022

Accepted 10.08.2022